PREDICTION OF FUTURE OBSERVATIONS AND ESTIMATION OF THE PARAMETERS IN STOCHASTIC REGRESSION MODELS

By

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To

My

Family

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CONTENTS

ABSTRACT

In this thesis, estimation of linear functions of the parameters in a stochastic regression model and the prediction of the vector of future observations related stochastically with a set of non-random variables, are considered. The minimum mean squared linear unbiased estimator and a class of shrinkage estimators of the parameters of interest are obtained. Properties of these estimators are examined analytically and numerically. However, analytic comparison of the estimators is not possible. Therefore, the estimators are compared by means of simulation. Our simulated results indicate that in the sense of mean squared error the shrinkage estimators are better than the minimum mean squared linear unbiased estimators.

CHAPTER 1

DESCRIPTION OF THE MODEL AND REVIEW OF THE LITERATURE

1.1 Introduction

Rao (1965), Swamy (1971), Maddala (1977, Chapter 17) Swamy and Mehta (1975), Harville (1976) and Pfeffermann (1984) studied stochastic regression models. Regression models with random coefficients were proposed mainly to characterize situations where the coefficients vary over certain domains. The coefficients may vary over time, across individuals strata, etc. Therefore, in some situations it becomes necessary to resort to the random coefficients models.

In this chapter we shall briefly describe different types of stochastic regression models that have been considered by a number of authors.

Pfeffermann (1984) considered the following general regression model with stochastic coefficients:

$$Y = X\,\beta + \varepsilon\,,$$
$$\beta = A\,\nu + v\,, \qquad (1.1.1)$$

where

Y is an (nx1) vector of observations,

X is an (nxp) matrix of known values,

β is a (px1) vector of unknown stochastic coefficients,

A is a (pxk) matrix of known values,

ν is a known or unknown fixed (kx1) vector,

ε is an (nx1) vector of random errors,

v is a (px1) vector of random disturbances,

and
$$E(\varepsilon) = 0 \quad , \quad E(\varepsilon\varepsilon') = \Sigma, \tag{1.1.2}$$
$$E(v) = 0 \quad , \quad E(vv') = \Delta \quad , \tag{1.1.3}$$
$$E(v\,\varepsilon') = 0 \tag{1.1.4}$$

Σ and Δ are known positive definite matrices.

Models in which the vectors of coefficients vary over time were considered by Duncan and Horn (1972), Rosenberg (1972) and Cooley and Prescot (1973). In a cross-sectional analysis employing regression models, it is sometimes appropriate to allow the vector of coefficients to vary over different clusters of units. Models in which the variation in the vectors of coefficients is over different subgroups of finite populations have been considered by Fay and Herriot (1979) and Rubin (1980). However, in this case the basic model consists of M distinct regression relations :

$$Y_i = X_i\,\beta_i + \varepsilon_i \qquad i = 1, 2, \ldots, M \tag{1.1.5}$$

where,

Y_i is the (N_ix1) vector of obervations drawn randomly from the i^{th} subgroup of the populaion,

X_i is the (N_ixk_o) matrix of input values,

β_i is the k_o-vector of unknown regression coefficients

and ε_i is the (N_ix1) vector of random errors.

It is easily seen, that models of this kind are special cases of the general model defined by (1.1.1) - (1.1.4). For this let

$$Y' = (Y_1' \quad Y_2' \ldots Y_M')$$

$$X = \begin{bmatrix} X_1 & 0 & \ldots & 0 \\ 0 & X_2 & \ldots & 0 \\ 0 & 0 & \ldots & X_M \end{bmatrix},$$

$$\beta' = \left(\beta_1' \; \beta_2' \quad \ldots \quad \beta_M' \right), \quad \text{and}$$

$$\varepsilon' = \left(\varepsilon_1' \; \varepsilon_2' \quad \ldots \quad \varepsilon_M' \right)$$

This shows that the models in (1.1.5) can be expressed in the form of the models described by (1.1.1) - (1.1.4).

Two models of this type that are often analysed in the literature are described below.

Model a: The vectors β_i, $i = 1, 2, \ldots, M$ in relation (1.1.5) can be considered as independent drawings from a multivariate distribution having $E(\beta_i) = \nu$ and $E(\beta_i-\nu)(\beta_i-\nu)' = R$. For this model, $A = [I_M \oplus I_{k_o}]$ where $\oplus$ denotes Kronecker product and $\Lambda = [I_M \oplus R]$.

Models of this kind are often used in econometrics to analyse time series and cross-sectional data (see Swamy 1971).

Model b: Furthermore assume that the vectors β_i, $i = 1, 2, \ldots, M$ in relation (1.1.5) may be generated by a Markovian structure of the form $\beta_{i+1} = P\,\beta_i + \nu_{i+1}$, $i = 0, 1, \ldots, (M-1)$, where P is a known $(k_o x k_o)$ transition matrix, ν_i's are independent random disturbances with variance covariance matrix R and β_o is a fixed starting state. This is a typical

Kalman filter model which usually appears in the engineering literature (see Sarris 1973).

1.2 Estimation of Linear Function of Parameters

Pfeffermann (1984) considered the problem of estimation of a linear function

$$W(\nu,\beta) = W_1' \nu + W_2' \beta , \tag{1.2.1}$$

where W_1 and W_2 are known fixed vectors, of the stochastic coefficient vector β and the fixed parameter vector ν.

He introduced the following notion:

Let $L'Y$ be an estimator of $W(\nu,\beta)$. An estimator $L'Y$ of $W(\nu,\beta)$ is said to be ξ-unbiased if

$$E_\xi\left[L'Y - W(\nu,\beta) \right] = 0 , \tag{1.2.2}$$

where ξ denotes the joint disribution of Y and β. The variance and the MSE of $L'Y$ are given by

$$Var(L'Y) = L'E_\xi\left[Y - E_\xi(Y) \right] \left[Y - E_\xi(Y) \right]' L$$

and

$$MSE(L'Y) = E_\xi\left[L'Y - W(\nu,\beta) \right] \left[L'Y - W(\nu,\beta) \right]'$$

respectively.

Further, an estimator $L_*'Y$ is said to be minimum mean squared error linear unbiased estimator (MMSLUE) of $W(\nu,\beta)$ if $L_*'Y$ is ξ-unbiased and the MSE of $L_*'Y$ is less than or equal to the MSE of any other linear ξ-unbiased estimator $L'Y$.

Swamy (1971) applied Gauss Markov theorem to obtain the MMSLUE's of $W_1'\nu$ and β.

Pfeffermann (1984) developed the MMSLUE of $W(\nu,\beta)$. We shall briefly discuss their findings in the following sections.

1.3 Optimal Estimation of $W_1'\nu$

To obtain an optimal estimator of $W_1'\nu$ Swamy, (1971) rewrote the model (1.1.1) as follows :

$$Y = X\,\beta + \varepsilon = X\,(A\nu + v) + \varepsilon$$

$$= X\,A\,\nu + u \qquad (1.3.1)$$

where $u = X\,v + \varepsilon$.

We observe from (1.1.1) - (1.1.4) that

$$E(u) = 0 \quad \text{and} \quad E(uu') = X\,\Delta\,X' + \Sigma = \Lambda \qquad (1.3.2)$$

Direct application of the Generalized Gauss Markov theorem produces the MMSLUE of $W_1'\nu$. This generalized least squares estimator of $W_1'\nu$ is given by

$$W_1'\hat{\nu} = W_1'\,(\,A'\,X'\,\Lambda^{-1}\,X\,A\,)^{-1}\,A'\,X'\,\Lambda^{-1}\,Y \qquad (1.3.3)$$

(also see Fisk (1967) and Swamy (1971)).

From (1.3.3) we have

$$E(W_1'\hat{\nu}) = W_1'\nu$$

and

$$E(W_1'(\hat{\nu}-\nu)(\hat{\nu}-\nu)'W_1) = W_1'[AX'\Lambda^{-1}XA]^{-1}A'X'\Lambda'\Lambda\Lambda^{-1}XA\,[A'x'\Lambda^{-1}XA]^{-1}W_1$$

$$= W_1'\,[A'\,X'\,\Lambda^{-1}\,X\,A\,]^{-1}\,W_1\,. \qquad (1.3.4)$$

Chipman (1964) and Rao (1965b, pp 192) obtained the MMSLUE of $W(\nu,\beta)=W_1'\nu+W_2'\beta$ when $E(\beta) = A\nu$ is known.

Their estimator of $W(\nu,\beta)$ is given by

$$\hat{W}(\nu,\beta) = W_1'\nu + W_2'A\nu + W_2'\Delta X'\Lambda^{-1}\,(Y-XA\nu) \qquad (1.3.5)$$

From (1.3.5) we get

$$E_\xi(\hat{W}(\nu,\beta)) = W_1' \nu + W_2' A \nu$$

and

$$E_\xi[\{\hat{W}(\nu,\beta) - W(\nu,\beta)\} \{\hat{W}(\nu,\beta) - W(\nu,\beta)\}']$$

$$= W_2'\Delta X'\Lambda^{-1} E_\xi[Y-XA\nu][Y-XA\nu]'\Lambda^{-1}X\Delta W_2 + W_2'E_\xi(vv')W_2$$

$$= W_2'\Delta X'\Lambda^{-1}\Lambda\Lambda^{-1}X\Delta W_2 + W_2'\Delta W_2$$

$$= W_2'\Delta X'\Lambda^{-1}X\Delta W_2 + W_2'\Delta W_2 \qquad (1.3.6)$$

1.4 Optimal Estimation of $W(\nu,\beta)$ when ν is unknown

Rao (1965a) derived the MMSLUE of $W_2'\beta$, assuming $A = I$. His estimator is given by

$$W_2' \hat{\beta} = W_2'(X'\Sigma^{-1}X)^{-1} X'\Sigma^{-1}Y = W_2' b \quad , \qquad (1.4.1)$$

where

$$b = (X' \Sigma^{-1} X)^{-1} X' \Sigma^{-1} Y.$$

Clearly,

$$E_\xi(b) = (X' \Sigma^{-1} X)^{-1} X' \Sigma^{-1} E(Y) = A \nu$$

and

$$E_\xi(b-A\nu)(b-A\nu)' = (X'\Sigma^{-1}X)^{-1} X'\Sigma^{-1}\Lambda\Sigma^{-1}X(X'\Sigma^{-1}X)^{-1}$$

$$= (X'\Sigma^{-1}X)^{-1} X'\Sigma^{-1}[X\Delta X'+\Sigma] \Sigma^{-1}X(X'\Sigma^{-1}X)^{-1}$$

$$= \Delta + (X'\Sigma^{-1}X)^{-1} .$$

Therefore,

$$E(W_2' \hat{\beta}) = W_2' A \nu$$

and

$$E(W_2'(b-A\nu)(b-A\nu)' W_2) = W_2'\Delta W_2 + W_2'(X'\Sigma^{-1}X)^{-1} W_2 \qquad (1.4.2)$$

Harville (1976) derived the MMSLUE of $W(\nu,\beta)$ allowing for any matrix A, as

$$\hat{W}(\nu,\beta) = W_1'\hat{\nu} + W_2'A\hat{\nu} + W_2'\Delta X'\Lambda^{-1}(Y-XA\hat{\nu}) \ , \qquad (1.4.3)$$

where

$$\hat{\nu} = (A'X'\Lambda^{-1}XA)^{-1} \ A'X'\Lambda^{-1}Y \ .$$

The estimator $\hat{W}(\nu,\beta)$ may be rewritten as

$$\hat{W}(\nu,\beta) = (W_1' + W_2'A - W_2'\Delta X'\Lambda^{-1}XA)\ \hat{\nu} + W_2'\Delta X'\Lambda^{-1}Y$$

Let

$$\left.\begin{aligned} k_1 &= (W_1' + W_2'A - W_2'\Delta X'\Lambda^{-1}XA) \\ k_2 &= (A'X'\Lambda^{-1}XA)^{-1} \ A'X'\Lambda^{-1} \\ k_3 &= W_2'\Delta X'\Lambda^{-1} \\ k_4 &= k_1 \ k_2 \ , \end{aligned}\right\} \qquad (1.4.4)$$

so

$$\hat{W}(\nu,\beta) = Y_4 \ Y + k_3 \ Y$$
$$= (k_4 + k_3) \ Y$$

and

$$E(\hat{W}(\nu,\beta)) = (k_4 + k_3) \ X \ A \ \nu$$

The dispersion matrix of $\hat{W}(\nu,\beta)$ is given by

$$E[(\hat{W}(\nu,\beta) - E(\hat{W}(\nu,\beta))]^2 = (k_4 + k_3) \ \Lambda \ (k_4 + k_3)' \ ,$$

where k_3 and k_4 are defined by (1.4.4) and Λ is defined by (1.3.2).

CHAPTER 2

ESTIMATION OF PARAMETERS AND PREDICTION OF FUTURE OBSERVATIONS FOR A GENERAL RANDOM COEFFICIENTS MODEL

2.1 Introduction

In this chapter, we shall introduce a more general model than the one considered by Pfeffermann (1984). Let

$$Y_1 = X_1 \beta_1 + e_1 \qquad (2.1.1)$$

$$Y_2 = X_2 \beta_2 + e_2 \qquad (2.1.2)$$

where Y_1 is an N-vector of observations,

Y_2 is an N_1-vector of unobserved future observations,

X_1 is an (NxP) design matrix of rank P,

X_2 is an $(N_1 x P_1)$ matrix of unknown values,

β_1 is a P-vector of unknown stochastic regression coefficients,

β_2 is a P_1-vector of unknown stochastic regression coefficient,

e_1 is an N-vector of random errors,

e_2 is an N_1-vector of random errors.

The vector of random errors $e' = (e_1' : e_2')$ is assumed to be distributed with

$$E(e_1) = 0 \quad , \quad E(e_2) = 0 \quad \text{and}$$

$$E\begin{bmatrix} e_1 e_1' & e_1 e_2' \\ e_2 e_1' & e_2 e_2' \end{bmatrix} = \sigma^2 \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}' & \Sigma_{22} \end{bmatrix} = \sigma^2 \Sigma \qquad (2.1.3)$$

where Σ is a known positive definite matrix and σ^2 is an unknown positive constant. Further, suppose that the stochastic vector $\beta' = (\beta_1' : \beta_2')$ satisfies the relation

$$\beta = A \nu + U, \qquad (2.1.4)$$

where $A' = (A_1' : A_2')$,

and A_1 is a $(P_1 x K)$ matrix of known values with rank K,

A_2 is a $(P_1 x K)$ known matrix with rank K,

ν is a K-dimensional known or unknown fixed vector,

$U' = (U_1' : U_2')$, where U_1 is a P-vector of random errors and U_2 is a P_1-vector of random errors.

We also assume that

$$E(U) = 0 \quad , \quad E(UU') = \sigma^2 \begin{bmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{12}' & \Delta_{22} \end{bmatrix} = \sigma^2 \Delta \qquad (2.1.5)$$

and

$$E(Ue') = \sigma^2 \begin{bmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{bmatrix} = \sigma^2 \delta \qquad (2.1.6)$$

where $e' = (e_1' : e_2')$, $U' = (U_1' : U_2')$, Δ is a known positive definite matrix and δ is a known matrix. δ may not be a symmetric matrix. Pfeffermann (1984) considered the above model with $\delta = 0$ and $\sigma^2 = 1$. The analysis of the data when $\delta \neq 0$ needs a special treatment. Moreover, Pfeffermann (1984) did not consider the problem of predicting the future observations vector when the parameters of the model are stochastic and the vectors of random errors are dependent. In this chapter, we are going to consider the following problems.

(*i*) Estimation of the linear parameteric function of the type

$$W(\nu,\beta) = W_1' \nu + W_2' \beta ,$$

where W_1 and W_2 are two given vectors with suitable dimensions, when ν is known and β is unknown.

(*ii*) Estimation of $W(\nu,\beta) = W_1'\nu + W_2'\beta$ when both ν and β are unknown.

(*iii*) Prediction of the future observations vector, Y_2.

In the following section we shall consider the problem stated in (*i*).

2.2 Estimation of $W(\nu,\beta) = W_1'\nu + W_2'\beta$, when ν is known

Theil (1963) considered the problem of estimation of the stochastic regression coefficients using a Bayesian approach. Rao (1965) and Pfeffermann (1984) followed the non-Bayesian (classical) approach to study the problem.

They assumed that all vectors of random errors are independently distributed. The regression analysis of data when the errors are correlated needs a special treatment, see Goldberger (1962).

We want to obtain the minimum mean squared linear unbiased estimator (MMSLUE) of $W(\nu,\beta) = W_1'\nu + W_2'\beta$ when the errors are correlated. Let

$$T(Y_1) = t_o + t_1' Y_1 ,$$

be a linear estimator of $W(\nu,\beta)$, where t_1 is a non-stochastic

N-vector and t_o is a scalar. The best choice of t_o and t_1 is given by the following theorem.

THEOREM 2.1

The minimum mean squared linear unbiased estimator of $W(\nu,\beta) = W_1'\nu + W_2'\beta$ is

$$T_*(Y_1) = t_o^* + t_1^{*'} Y_1$$

The mean squared error of $T_*(Y_1)$ is

$$MSE(T_*(Y_1)) = \sigma^2 T' \Delta T + 2\sigma^2 T' \delta D + \sigma^2 D' \Sigma D$$

where

$$t_1^* = Q^{-1} \left[X_1 H \Delta W_2 + H_1 \delta' W_2 \right] ,$$

$$t_o^* = (W_1' + W_2' A - t_1^{*'} X_1 A_1) \nu ,$$

$$Q = (X_1 H \Delta H' X_1' + 2 X_1 H \delta H_1' + H_1 \Sigma H_1') ,$$

$$H = \left[I_{pxp} : O_{pxp_1} \right] , \quad H_1 = \left[I_{NxN} : O_{NxN_1} \right] ,$$

$$T' = t_1^{*'} X_1 H - W_2 ,$$

$$D' = t_1^{*'} H_1 ,$$

X_1 is given in (2.1.1), A and A_1 are given in (2.1.4), Δ is defined by (2.1.5) and δ is given in (2.1.6).

Proof:

Let $T(Y_1) = t_o + t_1' Y_1$ be a linear estimator of $W(\nu,\beta) = W_1'\nu + W_2'\beta$. For $T(Y_1)$ to be unbiased, we must have

$$E_\xi [T(Y_1) - W(\nu,\beta)] = 0 , \qquad (2.2.2)$$

where ξ denotes the joint distribution of Y and β. Taking the expectation over ξ the joint distribution of Y and β, we get

$$E_\xi [T(Y_1) - W(\nu,\beta)] = t_o + t_1' X_1 A_1 \nu - W_1'\nu - W_2'A\nu .$$

From (2.2.2) we have

$$t_o + t_1' X_1 A_1\nu = W_1'\nu + W_2' A \nu$$

or

$$t_o = (W_1' + W_2'A - t_1'X_1A_1)\nu \qquad (2.2.3)$$

Now, the MSE of $T(Y_1)$ is given by

$$MSE(T(Y_1)) = E_\xi[T(Y_1) - W(\nu,\beta)]^2$$

$$= E_\xi[t_o + t_1' Y_1 - W_1' \nu - W_2'\beta]^2 \qquad (2.2.4)$$

Using (2.2.3), (2.1.1) and (2.1.4), (2.2.4) can be expressed as

$$MSE(T(Y_1)) = E_\xi\left(W_1'\nu + W_2'A\nu - t_1'X_1A_1\nu + t_1'Y_1 - W_1'\nu - W_2'\beta\right)^2$$

$$= E_\xi\left(W_2'A\nu - t_1'X_1A_1\nu + t_1'X_1 - W_2'\beta\right)^2$$

$$= E_\xi\left(W_2'A\nu - t_1'X_1A_1\nu + t_1'X_1(A_1\nu+U_1) + t_1'e_1 - W_2'A\nu - W_2'U\right)^2$$

$$= E_\xi\left(t_1'X_1U_1 + t_1'e_1 - W_2'U\right)^2 .$$

Suppose that $H = \left[I_{PxP} : O_{PxP_1}\right]$ and $H_1 = \left[I_{NxN} : O_{NxN_1}\right]$,

Then

$$MSE(T(Y_1)) = E_\xi\left(t_1X_1HU + t_1'H_1e - W_2'U\right)^2$$

$$= E_\xi\left((t_1'X_1H - W_2')U + t_1'H_1e\right)^2$$

$$= (t_1'X_1H - W_2')\, E(UU')\, (t_1'X_1H - W_2')' + 2(t_1'X_1H - W_2')\, E(Ue')\, H_1't_1 + t_1'H_1E(ee')\, H_1't_1$$

$$= \sigma^2t_1'X_1H\Delta H'X_1't_1 - 2\sigma^2t_1'X_1H\Delta W_2 + \sigma^2W_2'\Delta W_2 + 2\sigma^2t_1'X_1H\delta H_1't_1 - 2\sigma^2t_1'H_1\delta'W_2 + \sigma^2t_1'H_1\Sigma H_1't_1 \qquad (2.2.5)$$

By differentiating with respect to t_1, we get

$$\frac{\partial MSE(T(Y_1))}{\partial t_1} = 2\sigma^2X_1H\Delta H'X_1't_1 - 2\sigma^2X_1H\Delta W_2 + 4\sigma^2X_1H\delta H_1't_1$$

$$- 2\sigma^2 H_1 \delta' W_2 + 2\sigma^2 H_1 \Sigma H_1' t_1 .$$

Equating the derivative to zero, we get

$$2\sigma^2 X_1 H \Delta H' X_1' t_1 + 4\sigma^2 X_1 H \delta H_1' t_1 + 2\sigma^2 H_1 \Sigma H_1' t_1 = 2\sigma^2 H_1 \delta' W_2 + 2\sigma^2 X_1 H \Delta W_2$$

or,

$$\left[X_1 H \Delta H' X_1' + 2X_1 H \delta H_1' + H_1 \Sigma H_1'\right] t_1 = \left[X_1 H \Delta W_2 + H_1 \delta' W_2\right]$$

Let $Q = (X_1 H \Delta H' X_1' + 2X_1 H \delta H_1' + H_1 \Sigma H_1')$, and assuming Q to be positive definite matrix, we obtain

$$t_1^* = Q^{-1} (X_1 H \Delta W_2 + H_1 \delta' W_2) \tag{2.2.6}$$

Substituting (2.2.6) into (2.2.3) we get

$$t_o^* = \left[W_1' + W_2' A - t_1^{*\prime} X_1 A_1\right] \nu \tag{2.2.7}$$

Thus the MMSLUE of $W(\nu,\beta)$ becomes

$$T_*(Y_1) = t_o^* + t_1^{*\prime} Y_1 \tag{2.2.8}$$

where t_o^* and t_1^* are defined in (2.2.7) and (2.2.6) respectively.

The mean squared error of $T_*(Y_1)$ can be obtained by replacing the optimal values of t_o and t_1 given by (2.2.7) and (2.2.6) respectively into (2.2.4). This completes the proof.

2.3 Estimation of $W(\nu,\beta)$ when ν is unknown

In this section we shall estimate $W(\nu,\beta) = W_1'\nu + W_2'\beta$ when both ν and β are unknown, with ν fixed and β stochastic.

Let $L(Y_1) = \ell_o + \ell_1' Y_1$ be an estimator of $W(\nu,\beta)$, where ℓ_1 is a non-stochastic N-vector and ℓ_o is a constant.

THEOREM 2.3

For the case where ν is an unknown vector, the MMSLUE of $W(\nu,\beta)$ is given by

$$L_*(Y_1) = \ell_o^* + \ell_1^{*\prime} Y_1 ,$$

where

$$\ell_o^* = 0 , \quad \ell_1^* = Q^{-1}\left[MW_2 + X_1 A_1 R^{-1} Z\right],$$

$$Q = X_1 H \Delta H' X_1' + 2X_1 H \delta H_1' + H_1 \Sigma H_1' ,$$

$$M = (X_1 H \Delta + H_1 \delta') ,$$

$$R = A_1' X_1' Q^{-1} X_1 A_1 ,$$

$$Z = \left[W_1 + A' W_2 - A_1' X_1' Q^{-1} M W_2\right] ,$$

$$H = \left[I_{PxP} : O_{PxP_1}\right] \text{ and } H_1 = \left[I_{NxN} : O_{NxN_1}\right] .$$

The mean squared error of $L_*(Y_1)$ is

$$MSE(L_*(Y_1)) = \sigma^2 S' \Delta S + 2\sigma^2 S' \delta \psi + \sigma^2 \psi \Sigma \psi$$

where

$$S' = \ell_1^{*\prime} X_1 H - W_2' \quad \text{and} \quad \psi' = \ell_1^{*\prime} H_1 .$$

PROOF

For $L(Y_1) = \ell_o + \ell_1' Y_1$ to be unbiased we must have

$$E_\xi\left[L(Y_1) - W(\nu,\beta)\right] = 0 \qquad (2.3.1)$$

From (2.3.1) we get

$$\ell_o + \ell_1' X_1 A_1 \nu - W_1' \nu - W_2' A \nu = 0 \qquad (2.3.2)$$

It follows from (2.3.2) that

$$\ell_o = 0 \quad \text{and} \quad \ell_1' X_1 A_1 = (W_1' + W_2' A) . \qquad (2.3.3)$$

By definition we have

$$MSE(L(Y_1)) = E_\xi\left[\ell_1'X_1\beta_1 + \ell_1'e_1 - W_1'\nu - W_2'\beta\right]^2$$

$$= E_\xi\left[\ell_1'X_1(A_1\nu+U_1) + \ell_1'e_1 - W_1'\nu - W_2'A\nu - W_2'U\right]^2 .$$

Using (2.3.3) we get

$$MSE(L(Y_1)) = E_\xi\left[(\ell_1'X_1H - W_2')U + \ell_1'H_1e\right]^2$$

where $H = \left[I_{PxP} : O_{PxP_1}\right]$ and $H_1 = \left[I_{NxN} : O_{NxN_1}\right]$

Taking the expectation over the joint distribution of U and e, we obtain

$$MSE(L(Y_1)) = \sigma^2 \ell_1'X_1H\Delta H'X_1'\ell_1 - 2\sigma^2\ell_1'X_1H\Delta W_2 + \sigma^2W_2'\Delta W_2$$
$$+ 2\sigma^2\ell_1'X_1\delta H_1'\ell_1 - 2\sigma^2W_2'\delta H_1'\ell_1$$
$$+ \sigma^2\ell_1'H_1\Sigma H_1'\ell_1 . \qquad (2.3.4)$$

We now minimize the $MSE(L(Y_1))$ for the choice of ℓ_1 subject to the condition

$$\ell_1'X_1A_1 = (W_1' + W_2'A)$$

Let

$$F = MSE(L(Y_1)) - 2\sigma^2(\ell_1'X_1A_1 - W_1' - W_2'A)\lambda ,$$

where λ is a vector of Lagrange's multipliers, and $MSE(L(Y_1))$ is given by (2.3.4).

Now, diferentiating F with respect to ℓ_1 and with respect to λ, we get

$$\frac{\partial F}{\partial \ell_1} = 2\sigma^2X_1H\Delta H'X_1'\ell_1 - 2\sigma^2X_1H\Delta W_2 + 4\sigma^2X_1H\delta H_1'\ell_1$$
$$- 2\sigma^2H_1\delta'W_2 + 2\sigma^2H_1\Sigma H_1'\ell_1 - 2\sigma^2X_1A_1\lambda \qquad (2.3.5)$$

and

$$\frac{\delta F}{\partial \lambda} = -2\sigma^2(A_1'X_1'\ell_1 - W_1 - A'W_2) \qquad (2.3.6)$$

Then equating (2.3.5) and (2.3.6) to zero, we obtain

$$(X_1H\Delta H'X_1'+2X_1H\delta H_1'+H_1\Sigma H_1')\ell_1 = (X_1H\Delta+H_1\delta')W_2+X_1A_1\lambda \qquad (2.3.7)$$

and

$$A_1'X_1'\ell_1 = W_1 + A'W_2 \quad . \qquad (2.3.8)$$

Let $\quad Q = (X_1H\Delta H'X_1' + 2X_1H\delta H_1' + H_1\Sigma H_1')$ and

$\quad M = (X_1H\Delta + H_1\delta')$.

If Q is non-singular, then from (2.3.7) we have

$$\ell_1 = Q^{-1}MW_2 + Q^{-1}X_1A_1\lambda \qquad (2.3.9)$$

Pre-multiplying (2.3.9) by $A_1'X_1'$ we obtain

$$A_1'X_1'\ell_1 = A_1'X_1'Q^{-1}MW_2 + A_1'X_1'Q^{-1}X_1A_1\lambda \; .$$

Then from (2.3.8) we have

$$A_1'X'Q^{-1}MW_2 + A_1'X_1'Q^{-1}X_1A_1\lambda = W_1 + A'W_2$$

or,

$$A_1'X_1'Q^{-1}X_1A_1\lambda = (W_1 + A'W_2 - A_1'X_1'Q^{-1}MW_2) \qquad (2.3.10)$$

Assume that the matrix X_1A_1 is of full rank, therefore, $R = A_1'X_1'Q^{-1}X_1A_1$ is non-singular.

Hence, from (2.3.10) we get

$$\lambda = R^{-1}\left[W_1 + A'W_2 - A_1'X_1'Q^{-1}MW_2\right] \; . \qquad (2.3.11)$$

Substituting (2.3.11) into (2.3.9) we obtain

$$\ell_1^* = Q^{-1}MW_2 + Q^{-1}X_1A_1R^{-1}Z$$

$$= Q^{-1}\left[MW_2 + X_1A_1R^{-1}Z\right] \qquad (2.3.12)$$

where, $Z = (W_1 + A'W_2 - A_1'X_1'Q^{-1}MW_2)$.

Therefore, the MMSLUE of $W(\nu,\beta)$ when ν is unknown is given by

$$L_*(Y_1) = \left\{ Q^{-1}\left[MW_2 + X_1A_1R^{-1}Z\right] \right\}' \; Y_1 \qquad (2.3.13)$$

The MSE of $L_*(Y_1)$ can be obtained by substituting

(2.3.12) into (2.3.4). This gives

$$MSE(L_*(Y_1)) = \sigma^2 \ell_1^{*\prime} \left\{X_1 H\Delta H' X_1' + 2X_1 H\delta H_1' + H_1 \Sigma H_1'\right\} \ell_1^* + \sigma^2 W_2' \Delta W_2$$
$$- 2\sigma^2 \ell_1^{*\prime} X_1 H\Delta W_2 - 2\sigma^2 W_2' \delta H_1' \ell_1^*$$

Let $S' = \ell_1^{*\prime} X_1 H - W_2'$ and $\psi' = \ell_1^{*\prime} H_1$, then we obtain

$$MSE(L_*(Y_1)) = \sigma^2 (S + W_2)' \Delta (S + W_2) + 2\sigma^2 (S + W_2)' \delta\psi + \sigma^2 \psi' \Sigma \psi$$
$$- 2\sigma^2 (S + W_2)' \Delta W_2 - 2\sigma^2 W_2' \delta\psi + \sigma^2 W_2' \Delta W_2$$
$$= \sigma^2 S' \Delta S + 2\sigma^2 S' \delta\psi + \sigma^2 \psi' \Sigma \psi$$

This completes the proof.

Corollary 1:

The MMSLUE of $W_2'\beta$ is given by

$$\alpha(Y_1) = (MW_2 + X_1 A_1 R^{-1} Z_2)' \, Q'^{-1} Y_1$$

and the MSE of $\alpha(Y_1)$ is

$$MSE(\alpha(Y_1)) = \sigma^2 S_1 \Delta S_1 + 2\sigma^2 S_1' \delta \psi_1 + \sigma^2 \psi_1' \Sigma \psi_1$$

where,

$$S_1 = \alpha_*' X_1 H - W_2 \quad , \quad \psi_1' = \alpha_*' H_1 \ ,$$
$$\alpha_* = (MW_2 + X_1 A_1 R^{-1} Z_2)' \, Q'^{-1}$$
$$Z_2 = (A' - A_1' X_1' Q^{-1} M) \, W_2 \quad ,$$

Q, R and M are given in Theorem (2.3).

Corollary 2:

The MMSLUE of $W_1'\nu$ is

$$\gamma(Y_1) = W_1' \, (Q^{-1} X_1 A_1 R^{-1})' \, Y_1$$

and the MSE of $\gamma(Y_1)$ is

$$MSE(\gamma(Y_1)) = \sigma^2 S_2' \Delta S_2 + 2\sigma^2 S_2' \delta \psi_2 + \sigma^2 \psi_2' \Sigma \psi_2$$

where,

$$S_2' = (Q^{-1}X_1A_1R^{-1}W_1)' \; X_1H$$

$$\psi_2' = (Q^{-1}X_1A_1R^{-1}W_1)' \; H_1$$

Q, R and M are given in Theorem (2.3).

2.4 Prediction of Y_2

In this section we shall consider the problem of prediction of Y_2, the unobserved vector of future observations. Let η be an arbitrary $(N_1 x N)$ matrix and $\hat{Y}_2 = \eta Y_1$ be a predictor of Y_2.

The matrix η that makes $\hat{Y}_2$ unbiased and minimizes the MSE of $\hat{Y}_2$ is given by the following theorem.

THEOREM 2.4

The minimum mean squared unbiased predictor of Y_2 is

$$\hat{Y}_2^{*} = \eta^{*} \, Y_1 \qquad (2.4.1)$$

and the MSE of $\hat{Y}_2^{*}$ is

$$MSE(\hat{Y}_2^{*}) = \sigma^2 V'\Delta V + 2\,\sigma^2 V'\delta N + \sigma^2 N'\Sigma N \qquad (2.4.2)$$

where

$$\eta^{*} = \Big\{ \Big[J + (X_2A_2 - JX_1'Q^{-1}X_1A_1 + I_1H_1'Q^{-1}X_1A_1)$$

$$(A_1'X_1'Q^{-1}X_1A_1)^{-1} \; A_1'\Big] \; X_1' + I_1H_1' \Big\} \; Q^{-1} \; ,$$

$$Q = (X_1H\Delta H'X_1' + 2\; X_1H\delta H_1' + H_1\Sigma H_1') \; ,$$

$$J = (X_2H_2\Delta H' + H_3\delta'H') \; ,$$

$$I_1 = (X_2H_2\delta + H_3\Sigma) \; ,$$

$$V' = (\eta^{*}X_1H - X_2H_2) \; ,$$

$$N' = (\eta^{*}X_1 - H_3) \ ,$$

$$H = \left[I_{P\times P} \ \vdots \ 0_{P\times P_1}\right] \ , \qquad H_1 = \left[I_{N\times N} \ \vdots \ 0_{N\times N_1}\right] \ ,$$

$$H_2 = \left[I_{P_1\times P} \ \vdots \ 0_{P_1\times P_1}\right] \ \text{and} \qquad H_3 = \left[I_{N_1\times N} \ \vdots \ 0_{N_1\times N_1}\right] \ .$$

PROOF :

For $\hat{Y}_2$ to be unbiased we must have

$$E_{\xi}(\eta Y_1 - Y_2) = 0 \tag{2.4.3}$$

where ξ deontes the joint distribution of Y and β.

From (2.4.3) we get

$$\eta \ X_1 \ A_1 = X_2 \ A_2 \tag{2.4.4}$$

The MSE of $\hat{Y}_2$ is given by

$$MSE(\hat{Y}_2) = E_{\xi}(\eta Y_1 - Y_2)\,(\eta Y_1 - Y_2)' \tag{2.4.5}$$

We consider

$$\begin{aligned}\eta Y_1 - Y_2 &= \eta(X_1\beta_1 + e_1) - X_2\,\beta_2 - e_2 \\ &= \eta X_1(A_1\nu + U_1) + \eta e_1 - X_2(A_2\nu + U_2) - e_2 \\ &= \eta X_1 A_1\nu + \eta X_1 U_1 + \eta e_1 - X_2 A_2\nu - X_2 U_2 - e_2\end{aligned}$$

Using (2.4.4) we have

$$\begin{aligned}\eta Y_1 - Y_2 &= \eta X_1 U_1 - X_2 U_2 + \eta e_1 - e_2 \\ &= \eta X_1 H U - X_2 H_2 U + \eta H_1 e - H_3 e\end{aligned} \tag{2.4.6}$$

where $\quad H = \left[I_{P\times P} \ \vdots \ 0_{P\times P_1}\right] \ , \quad H_1 = \left[I_{N\times N} \ \vdots \ 0_{N\times N_1}\right] \ ,$

$\qquad H_2 = \left[I_{P_1\times P} \ \vdots \ 0_{P_1\times P_1}\right] \ \text{and} \ H_3 = \left[I_{N_1\times N} \ \vdots \ 0_{N_1\times N_1}\right]$

Then using (2.4.6) into (2.4.5) we get

$$\begin{aligned}MSE(\hat{Y}_2) = &(\eta X_1 H - X_2 H_2)\ E(UU')\ (\eta X_1 H - X_2 H_2)' \\ &+ (\eta X_1 H - X_2 H_2)\ E(Ue')(\eta H_1 - H_3)'\end{aligned}$$

$$+ (\eta H_1 - H_3)(E(e\nu')) (\eta X_1 H - X_2 H_2)'$$

$$+ (\eta H_1 - H_3) E(ee') (\eta H_1 - H_3)'$$

$$= \sigma^2 \eta X_1 H \Delta H' X_1' \eta' - \sigma^2 \eta X_1 H \Delta H_2' X_2' - \Delta^2 X_2 H_2 \delta H_1' X_1' \eta'$$

$$+ \sigma^2 X_2 H_2 \Delta H_2' X_2 + \sigma^2 \eta X_1 H \delta H_1' \eta' - \sigma^2 \eta X_1 H \delta H_3'$$

$$- \sigma^2 X_2 H_2 \delta H_1' \eta' + \sigma^2 X_2 H_2 \delta H_3' + \sigma^2 \eta H_1 \delta' H' X_1' \eta'$$

$$- \sigma^2 \nu H_1 \delta' H_2' X_2' - \sigma^2 H_3 \delta' H' X_1' \eta' + \sigma^2 H_3 \delta' H_2' X_2'$$

$$+ \sigma^2 \eta H_1 \Sigma H_1' \eta' - \sigma^2 \eta H_1 \Sigma H_3' - \sigma^2 H_3 \Sigma H_1' \eta'$$

$$+ \sigma^2 H_3 \Sigma H_3' \qquad (2.4.7)$$

Now, we minimize a linear functional of $MSE(\hat{Y}_2)$ for the choice of η subject to the condition $\eta X_1 A_1 = X_2 A_2$

Let $\qquad \phi = \text{tr } MSE(\hat{Y}_2) - 2\sigma^2 \text{tr}(A_1' X_1' \eta' - A_2' X_2')\ \mu$

where μ, a matrix stands for the Lagrange's multipliers and $\text{tr}(\Omega)$ denotes the trace of a square matrix Ω.

Using (2.4.7) we get

$$\phi = \sigma^2 \text{ tr } \eta X_1 H \Delta H' X_1' \eta' - 2\sigma^2 \text{ tr } X_2 H_2 \Delta H' X_1' \eta' + \sigma^2 \text{tr} X_2 H_2 \Delta H_2' X_2'$$

$$+ 2\sigma^2 \text{tr} \eta X_1 H \delta H_1' \eta' - 2\sigma^2 \text{tr} H_3 \delta' H' X_1' \eta' - 2\sigma^2 \text{tr} X_2 H_2 \delta H_1' \eta'$$

$$+ 2\sigma^2 \text{tr} X_2 H_2 \delta H_3' + \sigma^2 \text{tr} \eta H_1 \Sigma H_1' \eta' - 2\sigma^2 \text{tr} H_3 \Sigma H_1' \eta'$$

$$+ \sigma^2 \text{tr} H_3 \Sigma H_3' - 2\sigma^2 \text{tr}(A_1' X_1' \eta' - A_2' X_2')\ \mu$$

Differentiating ϕ w.r.t. η and μ we obtain

$$\frac{\partial \phi}{\partial \eta} = 2\sigma^2 \eta X_1 H \Delta H' X_1' - 2\sigma^2 X_2 H_2 \Delta H' X_1' + 4\sigma^2 X_1 H \delta H_1'$$

$$- 2\sigma^2 H_3 \delta' H' X_1' - 2\sigma^2 X_2 H_2 \delta H_1' + 2\sigma^2 \eta H_1 \Sigma H_1'$$

$$- 2\sigma^2 H_3 \Sigma H_1' - 2\sigma^2 \mu A_1' X_1' \qquad (2.4.8)$$

$$\frac{\partial \phi}{\partial \mu} = - 2\sigma^2 (\eta X_1 A_1 - X_2 A_2) \qquad (2.4.9)$$

Equating both derivatives to zero and solving for η and μ, we get

$$\eta(X_1 H\Delta H' X_1' + 2X_1 H\delta H_1' + H_1 \Sigma H_1') = X_2 H_2 \Delta H' X_1' + H_3 \delta' H' X_1' + X_2 H_2 \delta H_1' + H_3 \Sigma H_1' + \mu A_1' X_1' \qquad (2.5.10)$$

and

$$\eta X_1 A_1 = X_2 A_2 \qquad (2.4.11)$$

Let

$$Q = (X_1 H\Delta H' X_1' + 2X_1 H\delta H_1' + H_1 \Sigma H_1')$$

$$J = (X_2 H_2 \Delta H' + H_3 \delta' H')$$

$$I_1 = (X_2 H_2 \delta + H_3 \Sigma)$$

Then from (2.4.10) we get

$$\eta = ((J + \mu A_1')X_1' + I_1 H_1')Q^{-1} \qquad (2.4.12)$$

Post-multiplying (2.4.12) by $X_1 A_1$ we obtain

$$\eta X_1 A_1 = ((J + \mu A_1')X_1' + I_1 H_1')Q^{-1} X_1 A_1$$

From (2.4.11) we get

$$((J + \mu A_1')X_1' + I_1 H_1')Q^{-1} X_1 A_1 = X_2 A_2 \qquad (2.4.13)$$

Relation (2.4.13) gives

$$\mu = (X_2 A_2 - J X_1' Q^{-1} X_1 A_1 - I_1 H_1' Q^{-1} X_1 A_1)(A_1' X_1' Q^{-1} X_1 A_1)^{-1} \qquad (2.4.14)$$

Substituting (2.4.14) in (2.4.12) we obtain

$$\eta^{*} = \left\{\left[J + (X_2 A_2 - J X_1' Q^{-1} X_1 A_1 - I_1 H_1' Q^{-1} X_1 A_1)(A_1' X_1' Q^{-1} X_1 A_1)^{-1} A_1'\right] X_1' + I_1 H_1'\right\} Q^{-1} \qquad (2.4.15)$$

Relation (2.4.15) gives an optimum choice of η. Therefore, an optimum predictor of Y_2 is

$$\hat{Y}_2^{*} = \eta^{*} Y_1$$

Now, replacing η in (2.4.7) by η^{*} we get

$$MSE(\hat{Y}_2^{*}) = \sigma^2 V' \Delta V + 2\sigma^2 V' \delta N + \sigma^2 N' \Sigma N$$

where

$$V' = (\eta^{*}X_1H - X_2H_2) \quad \text{and} \quad N' = (\eta^{*}H_1 - H_3)$$

This completets the proof.

CHAPTER 3

SHRINKAGE APPROACH FOR ESTIMATING STOCHASTIC REGRESSION COEFFICIENTS

3.1 Introduction

Stein (1960), Sclove (1968), Hoerl and Kennard (1970), Obenchain (1978) and others introduced shrinkage approach to estimate the parameters of the linear regression models.

Shrinkage approach is used to reduce the variance of an estimator. But reduction in variance causes an increase of bias. However, under certain conditions the reduction in variance overpowers the increase in squared bias and hence, in terms of MSE, a shrinkage estimator dominates, under certain conditions, the usual least squares estimator.

To the best of our knowledge this well known approach has yet not been used to estimate the unknown stochastic regression coefficients. In this chapter, our aim is to introduce this approach to estimate a linear function of the random coefficients of a linear regression model. We have shown that in some situations, the shrinkage estimators of the random coefficients have smaller MSE than that of the minimum mean squared linear unbiased estimator.

3.2 Shrinkage Estimator of a Linear Function of Stochastic Coefficients

We consider the model defined in (2.1.1) - (2.1.6). For simplicity we assume that ν is known and $\sigma^2 = 1$. The case where ν is unknown is considered in section 3.3. Let

$$D_*(Y_1) = C\ T_*(Y) \qquad (3.2.1)$$

where C is an arbitrary constant and $T_*(Y)$ is defined by (2.2.8), be an estimator of $W(\nu,\beta) = W_1'\nu + W_2'\beta$, W_1 and W_2 are two given vectors.

Now we are going to study some properties of $D_*(Y_1)$.

THEOREM 3.1

For fixed C,

(*i*) $MSE(D_*(Y_1)) = MSE(T_*(Y_1)) + (C-1)^2\ E_\xi(T_*(Y_1))^2$

$$+ 2(C-1)\ E_\xi\left\{T_*(Y_1)(T_*(Y_1) - W(\nu,\beta))\right\}$$

(*ii*) The value of C that minimize. $MSE(D_*(Y_1))$ is

$$C^* = \frac{E_\xi(T_*(Y_1)\ W(\nu,\beta))}{E_\xi(T_*(Y_1))^2}$$

where,

$$E_\xi(T_*(Y_1))^2 = t_1^{*\prime}\ (X_1\Delta_{11}X_1' + 2X_1\delta_{11} + \Sigma_{11})\ t_1^* + \nu'(W_1' + W_2'A)'\ (W_1' + W_2'A)\ \nu\ ,$$

$$E_\xi(T_*(Y_1)\ W(\nu,\beta)) = t_1^{*\prime}\ [\ H_1(XA\nu\nu'A' + X\Delta + \delta') - X_1A_1\nu\nu'A'\ ]\ W_2 + \nu'(W_1'+W_2'A)'(W_1'+W_2'A)\nu\ ,$$

$$MSE(T_*(Y_1)) = T'\Delta\ T + 2T'SD + D'\Sigma D\ ,$$

$H_1 = \left[I_{N\times N} : O_{N\times N_1}\right]$, t_1^*, T, and D are defined in theorem 2.1 of section 2.2.

Proof:

By definition

$$MSE(D_*(Y_1)) = E_\xi(D_*(Y_1) - T_*(Y_1) + T_*(Y_1) - W(\nu,\beta))^2$$

$$= E_\xi(D_*(Y_1) - T_*(Y_1))^2 + E_\xi(T_*(Y_1) - W(\nu,\beta))^2$$

$$+ 2E_\xi\left\{(D_*(Y_1)-T_*(Y_1))(T_*(Y_1)-W(\nu,\beta))\right\} \quad (3.2.2)$$

Using (3.2.1) into (3.2.2) we obtain that

$$MSE(D_*(Y_1)) = MSE(T_*(Y_1)) + (C-1)^2\ E_\xi(T_*(Y_1))^2$$

$$+ 2(C-1)E_\xi\left\{T_*(Y_1)(T_*(Y_1) - W(\nu,\beta))\right\} \quad (3.2.3)$$

Now,

$$E_\xi(T_*(Y_1))^2 = Var(T_*(Y_1)) + (E_\xi(T_*(Y_1)))^2$$

$$= t_1'^*(X_1\Delta_{11}X_1' + 2X_1\delta_{11} + \Sigma_{11})\ t_1^*$$

$$+ \nu'(W_1' + W_2'A)'\ (W_1' + W_2'A)\nu \quad (3.2.4)$$

and

$$E_\xi(T_*(Y_1)W(\nu,\beta)) = E_\xi((t_o^* + t_1'^*Y_1)\ W(\nu,\beta))$$

$$= t_o^*(W_1'+W_2'A)\nu + E_\xi(t_1'^*Y_1(W_1'\nu+W_2'\beta))$$

$$= t_o^*(W_1'+W_2'A)\nu + t_1'^*X_1A_1\nu W_1'\nu$$

$$+ E_\xi(t_1'^*\ Y_1\ W_2'\beta)\ . \quad (3.2.5)$$

But,

$$E_\xi(t_1'^*Y_1W_2'\beta) = E_\xi(t_1'^*Y_1\beta'W_2)$$

$$= E_\xi(t_1'^*H_1Y\beta'W_2)\ , \quad (3.2.6)$$

where $H_1 = \left[I_{NxN}\ \vdots\ O_{NxN_1}\right]$.

From (2.1.1) and (2.1.2) we have

$$Y = X\beta + e\ , \quad \text{where} \quad X = \begin{bmatrix} X_1 & O_{NxP_1} \\ O_{N_1xP} & X_2 \end{bmatrix} \quad (3.2.7)$$

Then, using (3.2.7) into (3.2.6) we get

$$E_\xi(t_1'^{*}Y_1W_2'\beta) = t_1'^{*}H_1E_\xi((X\beta + e)\ \beta')\ W_2$$

$$= t_1'^{*}H_1E_\xi(X\beta\beta' + e\beta')W_2 \quad . \qquad (3.2.8)$$

But from (2.1.4) we have

$$E_\xi(\beta\beta') = E_\xi(A\nu + U)\ (A\nu + U)'$$

$$= A\ \nu\ \nu'\ A' + \Delta \qquad (3.2.9)$$

and also

$$E_\xi(e\beta') = E_\xi(e(A\nu + U)')$$

$$= E_\xi(eU') = \delta' \ . \qquad (3.2.10)$$

Hence from (3.2.8), (3.2.9) and (3.2.10) we obtain that

$$E_\xi(t_1'^{*}Y_1W_2\beta) = t_1'^{*}H_1(X\ A\ \nu\nu'\ A' + X\Delta + \delta')W_2 \qquad (3.2.11)$$

Substituting (3.2.11) into (3.2.8) we find

$$E_\xi(T_*(Y_1)W(\nu,\beta)) = t_o^{*}(W_1' + W_2'A)\nu + t_1'^{*}X_1A_1\nu W_1'\nu$$
$$+ t_1'^{*}H_1(XA\nu\nu'A' + X\Delta + \delta')W_2 \qquad (3.2.12)$$

Using (2.2.7) into (3.2.12) we get

$$E_\xi(T_*(Y_1)W(\nu,\beta)) = t_1'^{*}\Big\{H_1(XA\nu\nu'A' + X\Delta + \delta') - X_1A_1\nu\nu'A'\Big\}W_2 +$$
$$+ \nu'(W_1' + W_2'A)'\ (W_1' + W_2'A)\nu \qquad (3.2.13)$$

Now we minimize $MSE(D_*(Y_1))$ given in (3.2.3) for the choice of C. The optimal value of C that could minimize. $MSE(D_*(Y_1))$ is given by

$$C^{*} = \frac{E_\xi[T_*(Y_1)W(\nu,\beta)]}{E_\xi[T_*(Y_1)]^2} \qquad (3.2.14)$$

This completes the proof.

A comparison of MSE's of $D_*(Y_1)$ and $T_*(Y_1)$ is made in the following theorem.

THEOREM 3.2

$$MSE(D_*(Y_1)) \Big|_{C=C^*} \leq MSE(T_*(Y_1))$$

where C^* is defined by (3.2.14).

Proof:

In Theorem 3.1 we have proved that $MSE(D_*(Y_1))$ attains its minimum at $C = C^* \neq 1$, but if $C = 1$ then,

$$MSE(D_*(Y_1)) = MSE(T_*(Y_1))$$

hence,

$$MSE(D_*(Y_1)) \Big|_{C=C^*} \leq MSE(T_*(Y_1))$$

3.3 Shrinkage Estimator for $W(\nu,\beta)$ when ν is unknown

In this section we shall define another estimator for $W(\nu,\beta)$ when ν is unknown and $\sigma^2 = 1$. Let

$$G_*(Y_1) = a\, L_*(Y_1) \ , \qquad (3.3.1)$$

where, "a" is an arbitrary constant and $L_*(Y_1)$ is given by (2.3.13), be an estimator of $W(\nu,\beta) = W_1'\nu + W_2'\beta$. The optimal value of "a" is given by the following theorem.

THEOREM 3.3

For fixed "a" ,

(*i*) $MSE(G_*(Y_1)) = MSE(L_*(Y_1)) + (a-1)^2 E_\xi(L_*(Y_1))^2$

$$+\ 2(a-1)E_{\xi}\left\{L_{*}(Y_1)(L_{*}(Y_1)-W(\nu,\beta))\right\}$$

(*ii*) The value of "a" that minimizes $MSE(G_{*}(Y_1))$ is given by

$$a^{*} = \frac{\ell_1'^{*}\left\{X_1A_1\nu W_1'\nu + H_1(XA\nu\nu'A' + X\Delta + \delta')W_2\right\}}{\ell_1'^{*}(X_1\Delta_{11}X_1' + 2X_1\delta_{11} + \Sigma_{11})\ \ell_1^{*} + \nu'(W_1' + W_2'A)'(W_1'+W_2'A)\nu}$$

where ℓ_1^{*}, H_1 and $MSE(L_{*}(Y_1))$ are given in Theorem 2.3.

Proof:

By definition we have

$$MSE(G_{*}(Y_1)) = E_{\xi}(G_{*}(Y_1) - L_{*}(Y_1) + L_{*}(Y_1) - W(\nu,\beta))^2$$

$$= MSE(L_{*}(Y_1)) + (a-1)^2\ E_{\xi}(L_{*}(Y_1))^2$$

$$+\ 2(a-1)E_{\xi}\left\{L_{*}(Y_1)\ (L_{*}(Y_1) - W(\nu,\beta))\right\} \qquad (3.3.2)$$

But,

$$E_{\xi}(L_{*}(Y_1))^2 = Var(L_{*}(Y_1)) + [E_{\xi}(L_{*}(Y_1))]^2$$

$$= \ell_1'^{*}(X_1\Delta_{11}X_1' + 2X_1\delta_{11} + \Sigma_{11})\ell_1^{*}$$

$$+\ \nu'(W_1' + W_2'A)'(W_1' + W_2'A)\ \nu \qquad (3.3.3)$$

and so

$$E_{\xi}(L_{*}(Y_1)W(\nu,\beta)) = E_{\xi}(\ell_1'^{*}Y_1(W_1'\nu + W_2'\beta)) \qquad (3.3.4)$$

Using (3.2.11) into (3.3.4) we get

$$E_{\xi}(L_{*}(Y_1)W(\nu,\beta)) = \ell_1'^{*}X_1A_1\nu W_1'\nu + \ell_1'^{*}H_1(XA\nu\nu'A' + X\Delta + \delta')W_2 \quad (3.3.5)$$

The value of "a" that minimize $MSE(G_{*}(Y_1))$ given in (3.3.2), is

$$a^{*} = \frac{E_{\xi}(L_{*}(Y_1)W(\nu,\beta))}{E_{\xi}(L_{*}(Y_1))^2} \qquad (3.3.6)$$

Using (3.3.3) and (3.3.5) in (3.3.6) we get

$$a^{*} = \frac{\ell_1'^{*}\left\{X_1A_1\nu W_1'\nu+H_1(XA\nu\nu'A'+X\Delta+\delta')W_2\right\}}{\ell_1'^{*}(X_1\Delta_{11}X_1'+2X_1\delta_{11}+\Sigma_{11})\ell_1^{*}+\nu'(W_1'+W_2'A)'(W_1'+W_2'A)\nu} \qquad (3.3.7)$$

This completes the proof.

Note that the value of a^{*} given by (3.3.7) is not operational because it is a function of the unknown parameter vector ν. Thus, to make a^{*} operational we use the MNSLUE of ν which is given in Corollary 2, page (18).

This gives

$$\hat{a}^{*} = \frac{\ell_1'^{*}\left\{X_1A_1\hat{\nu}W_1'\hat{\nu}+H_1(XA\hat{\nu}\hat{\nu}'A'+X\Delta+\delta')W_2\right\}}{\ell_1'^{*}(X_1\Delta_{11}X_1'+2X_1\delta_{11}+\Sigma_{11})\ell_1^{*}+\hat{\nu}'(W_1'+W_2'A)'(W_1'+W_2'A)\hat{\nu}} \qquad (3.3.8)$$

Now, consider

$$\tilde{\theta}_{*}(Y_1) = \hat{a}^{*}\, L_{*}(Y_1) \qquad (3.3.9)$$

where $\hat{a}^{*}$ is given by (3.3.8). In the following section we shall study some properties of $\tilde{\theta}_{*}(Y_1)$

3.3A Some Properties of $\tilde{\theta}_{*}(Y_1)$

We have

$$\tilde{\theta}_{*}(Y_1) = \left\{\ell_1'^{*}X_1A_1\hat{\nu}\hat{\nu}'W_1\ell_1'^{*}Y_1 + \ell_1'^{*}H_1(XA\hat{\nu}\hat{\nu}'A'+X\Delta+\delta')W_2\ell_1'^{*}Y_1\right\}$$
$$\left\{\ell_1'^{*}(X_1\Delta_{11}X_1'+2X_1\delta_{11}+\Sigma_{11})\ell_1^{*}+\hat{\nu}'(W_1'+W_2'A)'(W_1'+W_2'A)\hat{\nu}\right\}^{-1} \qquad (3.3.10)$$

Let

$$
\left.\begin{aligned}
b &= \ell_1'^{*}(X_1\Delta_{11}X_1'+2X_1\delta_{11}+\Sigma_{11})\ell_1^{*} \quad , \\
M &= (W_1'+W_2'A)'(W_1'+W_2'A) \quad , \\
r &= \ell_1'^{*}X_1A_1 \quad , \quad f = W_1\ell_1'^{*} \quad , \\
c &= \ell_1'^{*}H_1XA \quad , \\
m &= A'W_2\ell_1'^{*} \quad , \\
d &= \ell_1'^{*}H_1(X\Delta + \delta')\, W_2\ell_1'^{*} \quad , \\
\hat{\nu} &= \psi\, Y_1 \quad , \\
\text{and} \quad \psi &= (Q^{-1}\, X_1\, A_1\, R^{-1})' \quad .
\end{aligned}\right\} \tag{3.3.11}
$$

Then substituting (3.3.11) into (3.3.10)

$$
\tilde{\theta}_*(Y_1) = \left\{ r\psi Y_1Y_1'\psi' fY_1 + C\psi Y_1Y_1'\psi' mY_1 + dY_1 \right\} \left\{ b + Y_1'\psi' M\psi Y_1 \right\}^{-1} \tag{3.3.12}
$$

But we have

$$
\left\{ b + Y_1'\psi' M\psi Y_1 \right\}^{-1} = \left\{ b(1 + \frac{1}{b}\, Y_1'\psi' M\psi Y_1) \right\}^{-1} = \frac{1}{b}\left[1 + \frac{1}{b}\, Y_1'\psi' M\psi Y_1 \right]^{-1} \tag{3.3.13}
$$

And now if $\left\{\frac{1}{b}\, Y_1'\psi' M\psi Y_1 \right\} < 1$, then from (3.3.13) we get,

$$
\left\{b + Y_1'\psi' M\psi Y_1\right\}^{-1} \approx \frac{1}{b}\left[1 - \frac{1}{b}\, Y_1'\psi' M\psi Y_1 \right] \tag{3.3.14}
$$

Then, using (3.3.14) into (3.3.12) and taking expectation, we find out

$$
\begin{aligned}
E_\xi(\tilde{\theta}_*(Y_1)) \approx{}& \frac{1}{b}\, E_\xi\left[r\psi Y_1Y_1'\psi' fY_1 + C\psi Y_1Y_1'\psi' mY_1 + dY_1\right] \\
&- \frac{1}{b^2}\, E_\xi\left[r\psi Y_1Y_1'\psi' fY_1Y_1'\psi' M\psi Y_1 + C\psi Y_1Y_1'\psi' mY_1Y_1'\psi' M\psi Y_1 \right. \\
&\left. + dY_1Y_1'\psi' M\psi Y_1\right]
\end{aligned} \tag{3.3.15}
$$

We assume that $Y_1 \sim N(\mu, T)$, where

$$\mu = X_1 A_1 \nu \quad \text{and} \quad T = (X_1 \Delta_{11} X_1' + 2X_1 \delta_{11} + \Sigma_{11}).$$

Let $\eta_1 = X_1 U_1 + e_1$ and $Y_1 = \mu + \eta_1$. Clearly

$$\eta_1 \sim N(O,T) \quad \text{and} \quad R_1 = T^{-\frac{1}{2}} \eta_1 \sim N(O,I). \text{ We may write}$$

$$Y_1 = \mu + T^{\frac{1}{2}} R_1 \ . \tag{3.3.16}$$

and using (3.3.16) into (3.3.15) we get,

$$E_\xi(\tilde{G}_*(Y_1)) \approx \frac{1}{b} E_\xi\Big\{r\psi(\mu+T^{\frac{1}{2}}R_1)(\mu+T^{\frac{1}{2}}R_1)'\psi' f(\mu+T^{\frac{1}{2}}R_1)$$

$$+ C\psi(\mu + T^{\frac{1}{2}}R_1)(\mu + T^{\frac{1}{2}}R_1)'\psi' m(\mu + T^{\frac{1}{2}}R_1) + d(\mu+T^{\frac{1}{2}}R_1)\Big\}$$

$$- \frac{1}{b^2} E_\xi\Big\{r\psi(\mu+T^{\frac{1}{2}}R_1)(\mu+T^{\frac{1}{2}}R_1)' f(\mu+T^{\frac{1}{2}}R_1)(\mu+T^{\frac{1}{2}}R_1)'$$

$$\psi' M\psi(\mu+T^{\frac{1}{2}}R_1) + C\psi(\mu+T^{\frac{1}{2}}R_1)\ (\mu+T^{\frac{1}{2}}R_1)' m$$

$$(\mu+T^{\frac{1}{2}}R_1)\ (\mu+T^{\frac{1}{2}}R_1)'\psi' M\psi\ (\mu+T^{\frac{1}{2}}R_1)$$

$$+ d\psi(\mu+T^{\frac{1}{2}}R_1)\ (\mu+T^{\frac{1}{2}}R_1)'\psi' M\psi(\mu+T^{\frac{1}{2}}R_1)\ \Big\} \tag{3.3.17}$$

Taking expectation term by term into (3.3.17), we obtain that

$$E_\xi(\tilde{G}_*(Y_1)) = \frac{r}{b}\Big[\ \psi\mu\mu'\psi' f\mu + \psi T\psi' f\mu + 2\mu \operatorname{tr} T\psi' f\Big]$$

$$+ \frac{c}{b}\Big[\psi\mu\mu'\psi' m\mu + \psi T\psi' m\mu + 2\mu \operatorname{tr} T\psi' m\Big] + \frac{d}{b}\ \mu$$

$$- \frac{r}{b^2}\Big\{\psi\mu\mu'\psi' f\mu\mu'\psi' M\psi\mu + \psi\mu\mu'\psi' fT\psi' M\psi\mu$$

$$+ 4\mu\mu'\psi'\mu\psi\mu \operatorname{tr}\psi' fT + \psi T\psi' f\mu\mu'\psi' M\psi\mu$$

$$+ 2\psi T\psi' fT\psi' M\psi\mu + \psi T\psi' M\psi\mu \operatorname{tr} T\psi' f$$

$$+ 2\mu(2\operatorname{tr} T\psi' fT\psi' M\psi + \operatorname{tr} T\psi' f\ \operatorname{tr} T\psi' M\psi)$$

$$+ 2\psi T\psi' M\psi T\psi' f\mu + \psi T\psi' f\mu\ \operatorname{tr}\psi' M\psi T\Big\}$$

$$- \frac{c}{b^2} \Big\{ \psi\mu\mu'\psi' m\mu\mu'\psi' M\psi\mu + \psi\mu\mu'\psi' m T\psi'\mu\psi\mu$$

$$+ 4\mu\mu'\psi'\mu\psi\mu tr\psi' mT + \psi T\psi' m\mu\mu'\psi'\mu\psi\mu$$

$$+ 2\psi T\psi' mT\psi' M\psi\mu + \psi T\psi' M\psi\mu tr T\psi' m$$

$$+ 2\mu(2trT\psi' mT\psi' M\psi\psi\mu + trT\psi' m trT\psi' M\psi)$$

$$+ 2\psi T\psi' M\psi T\psi' m\mu + \psi T\psi' m\mu tr\psi' M\psi T \Big\}$$

$$- \frac{d}{b^2} \Big\{ \psi\mu\mu'\psi' M\psi\mu + \psi T\psi' M\psi\mu + 2\psi\mu trT\psi' M\psi \Big\} \qquad (3.3.18)$$

The expression which gave the MSE of $\hat{\theta}_*(Y_1)$ is very complicated. And so we shall compare $MSE(\hat{\theta}_*(Y_1))$ with $MSE(L_*(Y_1))$ by means of simulation.

3.4 Shrinkage Approach in Prediction

In Section 2.4 we have obtained MMSU predictor of Y_2, which is $\hat{Y}_2^* = \eta^* Y_1$, where η^* given in (2.4.18).

In this section we shall consider another predictor for Y_2 which is given by,

$$\psi_* = B\,\eta^*\,Y_1 \qquad (3.4.1)$$

where B is an $(N_1 x N)$ matrix.

The matrix B that minimize $trMSE(\psi_*)$ is given by the next theorem.

THEOREM 3.4

For a fixed matrix B,

$$MSE(\psi_*) = (B - I)\,E_\xi(\hat{Y}_2^*\,\hat{Y}_2^{*\prime})\,(B - I)' + MSE(\hat{Y}_2^*)$$
$$+ (B - I)\,E(\hat{Y}_2^*\,(\hat{Y}_2^* - Y_2)') + E_\xi((\hat{Y}_2^* - Y_2)\hat{Y}_2^{*\prime})(B - I)'$$

The matrix B that minimize $trMSE(\psi_*)$ is given by

$$B^* = \left[\eta^* (X_1(A_1\nu\nu'A_1' + \Delta_{11})X_1' + 2X_1\delta_{11} + \Sigma_{11}) \eta'^* \right]^{-1}$$

$$\left[X_1(A_1\nu\nu'A_2' + \Delta_{12})X_2' + X_1\delta_{12} + \delta_{21}'X_2' + \Sigma_{12} \right]' \eta'^* ,$$

where η^* is defined in (2.4.18).

Proof:

By definition we have

$$MSE(\psi_*) = E_\xi(\psi_* - \hat{Y}_2^* + \hat{Y}_2^* - Y_2)(\psi_* - \hat{Y}_2^* + \hat{Y}_2^* - Y_2)'$$

$$= E_\xi(\psi_* - \hat{Y}_2^*)(\psi_* - \hat{Y}_2^*)' + E_\xi(\psi_* - \hat{Y}_2^*)(\hat{Y}_2^* - Y_2)'$$

$$+ E_\xi(\hat{\psi}_*^*-Y_2)(Y_*-\hat{Y}_2^*)'+E_\xi(\hat{Y}_2^*-Y_2)(\hat{Y}_2^*-Y_2)' \qquad (3.4.2)$$

Substituting (3.4.1) into (3.4.2) we get

$$MSE(\psi_*) = (B-I)E_\xi(\hat{Y}_2^*\hat{Y}_2^{*\prime})(B-I)'+(B-I)E_\xi(\hat{Y}_2^*(\hat{Y}_2^*-Y_2)')$$

$$+ E_\xi((\hat{Y}_2^* - Y_2)\hat{Y}_2^{*\prime})'(B-I)' + MSE(\hat{Y}_2^*) \qquad (3.4.3)$$

Further,

$$E(\hat{Y}_2^*\ \hat{Y}_2^{*\prime}) = \eta^* E(Y_1Y_1')\ \eta'^*$$

$$= \eta^*(X_1(A_1\nu\nu'A_1'+\Delta_{11})X_1'+2X_1\delta_{11}+\Sigma_{11})\eta'^* , \qquad (3.4.4)$$

and

$$E(\hat{Y}_2^*\ Y_2') = \eta^*E(Y_1Y_2')$$

$$= \eta^*(X_1(A_1\nu\nu'A_2'+\Delta_{12})\ X_2'+X_1\delta_{12}+\delta_{21}'X_2'+\Sigma_{12}) \qquad (3.4.5)$$

Relations (3.4.4) and (3.4.5) may be used to simplify the expresion for the MSE of ψ_*. It is obvious from (3.4.3) that the matrix B that minimize $trMSE(\psi_*)$ is given by

$$B^* = \left[\eta^*(X_1(A_1\nu\nu'A_1' + \Delta_{11})X_1' + 2X_1\delta_{11} + \Sigma_{11})\ \eta'^*\right]^{-1}$$

$$\left[X_1(A_1\nu\nu'A_2' + \Delta_{12})X_2' + X_1\delta_{12} + \delta_{21}'X_2' + \Sigma_{12}\right]'\eta'^* \qquad (3.4.6)$$

THEOREM 3.6

If $B = B^{*}$, then

$$\mathrm{trMSE}(\psi_{*})\Big|_{B=B^{*}} \le \mathrm{trMSE}(\hat{Y}_2^{*})$$

where $\mathrm{tr}(\Omega)$ denotes the trace of a square matrix Ω.

Proof:

In theorem 3.4 we proved that $\mathrm{trMSE}(\psi_{*})$ attains its minimum at $B = B^{*}$. Note that if $B=I$ then $\mathrm{trMSE}(\psi_{*})=\mathrm{trMSE}(\hat{Y}_2^{*})$. But $B_{*} \ne I$. Therefore,

$$\mathrm{trMSE}(\psi^{*})\Big|_{B=B^{*}} \le \mathrm{trMSE}(\hat{Y}_2^{*})$$

This completes the proof.

CHAPTER 4

SIMULATION

4.1 Introduction

In Chapter 3 we proposed shrinkage estimators for a linear function of the stochastic parameters.

In this chapter we compare using simulation the MSE's of the shrinkage estimators and the estimators developed in Chapter 2.

As a matter of fact, the simulated results of the experiments are used to estimate the MSE's of the estimators and to estimate the shrinkage constants.

A description of the simulated experiment is given in the next section.

4.2 Simulation of the Experiments

First, we assume that $\sigma^2 = 1$ and $e \sim N(0, I_{n\times n})$, where $n = N+N_1$. We generate n random values from N(0,1), these random values are denoted by $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$, i.e. $e' = (\varepsilon_1 \ldots \varepsilon_n)$. Then for a given qxn matrix δ, a q-vector of random errors is computed using the relation

$$U = \delta e$$

where $q = p + p_1$.

Then, for a given k-vector of fixed values ν, a (qxk)-matrix of known values A, a q-vector β of stochastic coefficients is computed using the relation

$$\beta = A \nu + e$$

After this, for a given (Nxp) data matrix X_1, an N-vector of observations Y_1 is computed using the relation

$$Y_1 = X_1 \beta_1 + e_1$$

where, $\beta' = (\beta_1' : \beta_2')$ and $e' = (e_1' : e_2')$

Then for a given (k+q)-vector $W' = (W_1' : W_2')$ we calculate $W(\nu,\beta)$ using the relation

$$W(\nu,\beta) = W_1' \nu + W_2' \beta .$$

When ν is known, the estimator $T_*(Y_1)$ is computed using the relation

$$T_*(Y_1) = t_o^* + t_1^{*\prime} Y_1$$

where t_o^* and t_1^* are calculated using the relations given in (2.2.7) and (2.2.6) respectively. The shrinkage estimator $D_*(Y_1)$ is calculated using the relation

$$D_*(Y_1) = C^* T(Y_1)$$

where C^* is estimated by taking the average of the 600 sample values of

$$\frac{W(\nu,\beta)\ T_*(Y_1)}{[T_*(Y_1)]^2}$$

For the case when ν is unknown, the estimator $L_*(Y_1)$ is calculated using the relation

$$L_*(Y_1) = \ell_1^{*\prime} Y_1$$

where ℓ_1^* is computed from the relation (2.3.12).

For the shrinkage estimator, we obtain $\hat{\nu}$ using the relation

$$\hat{\nu} = \left[Q^{-1} x_1 A_1 R^{-1} \right]'$$

where Q and R are defined in theorem 2.3, $\hat{W}(\nu,\beta)$ is calculated using

$$\hat{W}(\nu,\beta) = W_1' \hat{\nu} + W_2' \beta$$

The shrinkage factor $\hat{a}_*$ is computed by taking the average of the 500 sample values of

$$\frac{\hat{W}(\nu,\beta) \; L_*(Y_1)}{[L_*(Y_1)]^2}$$

Hence, $\hat{\delta}_*(Y_1)$ is computed from the relation

$$\hat{\delta}_*(Y_1) = \hat{a}_* \; L_*(Y_1)$$

The MSE's of the above estimators are calculated by taking the averages of 500 sample values of each squared error $\left[\text{e.g.} \quad (T_*(Y_1) - W(\nu,\beta))^2 \right]$

The results are given in the following tables.

Table 4.1

Estimated Mean Squared Errors of $T_*(y_1)$ and $D_*(y_1)$

$p_1 = 1 \quad p = 1 \quad N_1 = 3 \quad N = 3 \quad n = 6 \quad q = 2 \quad k = 2$

$x_1' = [\ 1\ \ -1\ \ \ 2\ ] \qquad \delta = \begin{bmatrix} 1 & 3 & 1 & 0 & 0 & 0 \\ 0 & -2 & -1 & 0 & 3 & 1 \end{bmatrix}$

$W' = [\ 1\ \ 1\ \ 1\ \ 1\ ] \qquad A = \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}$

THE VECTOR $\underline{\mu}$	$MSE(T_*(Y_1))$	C^*	$MSE(D_*(Y_1))$
[0 0]´	11.92910	1.214600	11.90660
[1 0]´	12.04840	0.985397	12.03500
[.1 0]´	11.97000	0.970399	11.96880
[.2 .1]´	14.28400	1.318490	13.80340
[1 2]´	11.61570	0.993186	11.60070
[.5 .6]´	11.78070	0.938914	11.57930
[2.1 .7]´	15.21480	0.983385	15.10270
[0 -.1]´	10.93790	0.628592	10.82640
[0 -1]´	10.23820	1.020600	10.22710
[0 -3]´	11.19910	0.976844	11.07720
[-1 1]´	11.27360	0.805413	10.90150
[-1 2]´	10.32730	1.190090	10.18080
[-2 3]´	11.78750	1.123670	11.76320
[-1 -1]´	08.82317	0.968209	08.65116
[-.1 -.1]´	07.09393	1.292860	06.88071
[.05 .05]´	11.65200	0.721687	11.55590
[-.01-.01]´	12.28050	0.816637	12.25970
[2 -1]´	11.46140	0.985331	11.43500
[.9 .8]´	12.27310	0.978129	12.21460
[-.2 -.4]´	12.83180	1.106800	12.68110
[4 3]´	12.94060	1.007000	12.83200
[17 1]´	07.74278	1.002590	07.60916
[3 19]´	11.36010	1.002540	11.26830

Table 4.2

Estimated Mean Squared Errors of $T_*(y_1)$ and $D_*(y_1)$

$P_1 = 1 \quad p = 1 \quad N = 3 \quad N_1 = 3 \quad n = 6 \quad q = 2 \quad k = 2$

$X_1' = [1 \quad -1 \quad 2] \qquad \delta = \begin{bmatrix} 1 & 3 & 1 & 0 & 0 & 0 \\ 0 & -2 & -1 & 0 & 3 & 1 \end{bmatrix}$

$\nu' = [1 \quad 1] \qquad W_2' = [1 \quad 1] \qquad A = \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}$

THE VECTOR w_1	$MSE(T_*(Y_1))$	C^*	$MSE(D_*(Y_1))$
[1 2]′	12.38070	1.009980	12.36090
[1 3]′	11.88960	1.002600	11.88810
[1 5]′	11.29240	1.005760	11.28270
[2 3]′	11.32820	1.019320	11.23190
[2 4]′	13.06380	0.997004	13.06110
[−1 −1]′	10.87910	1.019760	10.84730
[−1 5]′	11.32760	0.989485	11.30280
[−1 10]′	08.97206	0.996081	08.96590
[−5 2]′	12.20200	0.990873	12.19650
[3 −1]′	10.43150	0.995545	10.42820
[7 2]′	12.69800	0.962515	12.13610
[9 1]′	10.39010	0.988871	10.33610
[9 3]′	11.33270	1.002990	11.32790
[3 8]′	15.43750	1.012890	15.35700

Table 4.3

Estimated Mean Squared Errors of $T_*(y_1)$ and $D_*(y_1)$

$P_1 = 1 \quad P = 1 \quad N = 3 \quad N_1 = 3 \quad n = 6 \quad q = 2 \quad k = 2$

$X_1' = [1 \quad -1 \quad 2] \qquad \delta = \begin{bmatrix} 1 & 3 & 1 & 0 & 0 & 0 \\ 0 & -2 & -1 & 0 & 3 & 1 \end{bmatrix}$

$\nu' = [1 \quad 1] \qquad W_1' = [1 \quad 1] \qquad A = \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}$

THE VECTOR W_2	$MSE(T_*(Y_1))$	C^*	$MSE(D_*(Y_1))$
[1 2]′	40.35430	1.061080	38.99870
[1 3]′	102.5960	1.019630	102.3530
[1 5]′	264.5600	0.986482	264.3150
[2 3]′	77.03230	0.977331	76.56790
[2 4]′	190.7300	1.050740	187.4520
[−1 −1]′	11.12490	0.906423	10.39200
[−1 5]′	475.5910	1.042800	474.1790
[−1 10]′	1353.840	0.980148	1352.520
[−5 2]′	487.5960	1.047490	487.3110
[3 −1]′	199.3610	0.918465	198.5430
[7 2]′	346.9580	1.030570	344.7400
[9 1]′	892.5490	1.040230	887.9640
[9 3]′	627.3470	0.983486	626.1960
[3 8]′	852.7930	0.957146	844.9690

Table 4.4

Estimated Mean Squared Errors of $T_*(y_1)$ and $D_*(y_1)$

$P_1 = 1 \quad P = 1 \quad N = 3 \quad N_1 = 3 \quad n = 6 \quad q = 2 \quad k = 2$

$X_1' = [1 \quad -1 \quad 2] \qquad \delta = \begin{bmatrix} 1 & 3 & 1 & 0 & 0 & 0 \\ 0 & -2 & -1 & 0 & 3 & 1 \end{bmatrix}$

$\nu' = [1 \quad 1] \qquad W' = [1 \quad 1 \quad 1 \quad 1]$

THE MATRIX A	$MSE(T_*(Y_1))$	C^*	$MSE(D_*(Y_1))$
$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$	08.53214	1.060930	08.39629
$\begin{bmatrix} -2 & -1 \\ 0 & 2 \end{bmatrix}$	11.88260	0.898418	11.86940
$\begin{bmatrix} -4 & 6 \\ -8 & 2 \end{bmatrix}$	10.93880	1.107990	10.92240
$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$	11.84180	1.092190	11.80480
$\begin{bmatrix} 2 & -7 \\ 11 & 1 \end{bmatrix}$	10.47120	1.002320	10.47080
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	10.41290	1.061020	10.38210

Table 4.5

Estimated Mean Squared Errors of $T_*(y_1)$ and $D_*(y_1)$

$P_1 = 1$ $P = 1$ $N = 3$ $N_1 = 3$ $n = 6$ $q = .2$ $k = 2$

$$A = \begin{bmatrix} 8 & -6 \\ -3 & 2 \end{bmatrix} \qquad \delta = \begin{bmatrix} 1 & 3 & 1 & 0 & 0 & 0 \\ 0 & -2 & -1 & 0 & 3 & 1 \end{bmatrix}$$

$\nu' = [1 \ 1]$ $W' = [1 \ 1 \ 1 \ 1]$

THE MATRIX X_1	$MSE(T_*(Y_1))$	C^*	$MSE(D_*(Y_1))$
[-2 1 3]'	1061.66	5.45183	895.030
[2 1 5]'	1019.92	6.27095	757.176
[6 2 4]'	821.783	6.60655	538.668
[2 1 1]'	1308.70	9.37803	538.141
[100 100 100]'	1141.69	8.40135	541.078
[-4 -7 -9]'	1393.05	8.78183	741.033
[-5 3 6]'	1201.37	7.08882	882.932
[.5 .5 .5]'	1055.76	8.62194	478.039

Table 4.6

Estimated Mean Squared Errors of $T_*(y_1)$ and $D_*(y_1)$

$P_1 = 1 \quad P = 1 \quad N = 3 \quad N_1 = 3 \quad n = 6 \quad q = 2 \quad K = 2$

$X_1' = [\,1 \quad -1 \quad 2\,] \qquad A = \begin{bmatrix} 8 & -6 \\ -3 & 2 \end{bmatrix}$

$\nu' = [\,1 \quad 1\,] \qquad W' = [\,1 \quad 1 \quad 1 \quad 1\,]$

$$\delta = \begin{bmatrix} -2 & 1 & 5 & 0 & 0 & 0 \\ 0 & -2 & -1 & 0 & 3 & 1 \end{bmatrix}$$

$MSE(T_*(Y_1)) = 2098.58$

$C^* = 7.10922$

$MSE(D_*(Y_1)) = 1452.31$

$$\delta = \begin{bmatrix} -2 & 1 & 5 & 0 & 0 & 0 \\ 0 & 5 & -1 & 0 & 7 & 6 \end{bmatrix}$$

$MSE(T_*(Y_1)) = 1966.99$

$C^* = 2.68094$

$MSE(D_*(Y_1)) = 1883.78$

$$\delta = \begin{bmatrix} 6 & 3 & 9 & 0 & 0 & 0 \\ 0 & 5 & -1 & 0 & 7 & 6 \end{bmatrix}$$

$MSE(T_*(Y_1)) = 9753.81$

$C^* = 3.27727$

$MSE(D_*(Y_1)) = 9257.89$

$$\delta = \begin{bmatrix} 6 & 3 & 9 & 0 & 0 & 0 \\ 0 & 10 & 11 & 1 & -9 & 3 \end{bmatrix}$$

$MSE(T_*(Y_1)) = 7551.96$

$C^* = 2.41351$

$MSE(D_*(Y_1)) = 6937.39$

Table 4.7

Estimated Mean Squared Errors for $T_*(y_1)$ and $D_*(y_1)$

$P_1 = 1 \quad P = 1 \quad N = 10 \quad N_1 = 10 \quad n = 20 \quad q = 2 \quad K = 2$

$X_1' = [1 \;\; 2 \; -1 \;\; 3 \;\; 2 \;\; 0 \; -6 \;\; 9 \;\; 5 \;\; 0] \qquad W' = [1 \; 1 \; 1 \; 1]$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \delta = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

THE VECTOR ν	$MSE(T_*(Y_1))$	C^*	$MSE(D_*(Y_1))$
[0 0]′	17.05840	1.027720	17.03810
[1 0]′	24.03930	1.064190	23.89430
[.1 0]′	16.31220	1.000860	16.31220
[.2 .1]′	16.95140	0.912940	16.74550
[1 2]′	18.81630	0.962672	18.72290
[.5 .6]′	17.79040	1.111220	17.29700
[2.1 .7]′	20.16350	1.007410	20.15970
[0 −.1]′	18.41010	1.015690	18.40300
[0 −1]′	14.88380	1.028530	14.85670
[0 −3]′	18.98230	1.071430	18.63250
[−1 1]′	21.33610	0.963053	21.29510
[−1 2]′	18.36260	1.089680	18.13540
[−2 3]′	19.27730	0.859015	18.65330
[−1 −1]′	17.71230	0.941053	17.55910
[−.1 −.1]′	19.39590	1.045860	19.38180
[.05 .05]′	17.55570	1.053870	17.47500
[−.01−.01]′	18.33170	1.066620	18.20070
[2 −1]′	17.31740	1.004740	17.31670
[.9 .8]′	18.63430	1.073730	18.43520
[−.2 −.4]′	22.99890	1.040830	22.95240
[4 3]′	21.90790	1.028070	21.75680
[17 1]′	19.52150	1.001830	19.51720

Table 4.8

Estimated Mean Squared Errors for $T_*(y_1)$ and $D_*(y_1)$

$P_1 = 1 \quad P = 1 \quad N = 10 \quad N_1 = 10 \quad n = 20 \quad q = 2 \quad K = 2$

$X_1' = [1 \;\; 2 \; -1 \;\; 3 \;\; 2 \;\; 0 \; -6 \;\; 9 \;\; 5 \;\; 0] \quad W_2' = [1 \;\; 1] \quad \nu' = [1 \;\; 1]$

$$A=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \quad \delta=\begin{bmatrix}1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\ -1&-1&-1&-1&-1&-1&0&0&1&1&1&1&1&1&1&1&1&1&1&1\end{bmatrix}$$

THE VECTOR W_1	$MSE(T_*(Y_1))$	C^*	$MSE(D_*(Y_1))$
[1 2]'	19.03230	0.957171	18.94010
[1 3]'	17.23350	0.994280	17.23140
[1 5]'	17.26210	1.026850	17.19820
[2 3]'	21.09090	1.005360	21.08820
[2 4]'	16.57770	0.927280	16.01650
[-1 -1]'	18.20440	0.982366	18.19720
[-1 5]'	19.22420	0.992711	19.22140
[-1 10]'	17.11940	0.988678	17.10140
[-5 2]'	18.15000	1.047000	18.08880
[3 -1]'	22.42530	1.060750	22.27990
[7 2]'	23.52410	0.925206	22.69960
[9 1]'	17.04150	0.999516	17.04150
[9 3]'	19.63490	0.962655	19.31250
[3 8]'	27.86930	1.013750	27.83850

Table 4.9

Estimated Mean Squared Errors for $T_*(y_1)$ and $D_*(y_1)$

$P_1 = 1 \quad P = 1 \quad N = 10 \quad N_1 = 10 \quad n = 20 \quad q = 2 \quad K = 2$

$X_1' = [1 \;\; 2 \;\; -1 \;\; 3 \;\; 2 \;\; 0 \;\; -6 \;\; 9 \;\; 5 \;\; 0] \qquad W_1' = [1 \;\; 1] \qquad \nu' = [1 \;\; 1]$

$$A=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \qquad \delta=\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\ -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\end{bmatrix}$$

THE VECTOR W_2	$MSE(T_*(Y_1))$	C^*	$MSE(D_*(Y_1))$
[1 2]´	38.50330	1.009100	38.49670
[1 3]´	79.84350	0.918838	78.62490
[1 5]´	209.5090	1.002200	209.5080
[2 3]´	122.9570	1.000100	122.9570
[2 4]´	131.2740	0.994138	131.2600
[-1 -1]´	18.70990	0.918774	18.56190
[-1 5]´	139.7850	1.046220	139.1560
[-1 10]´	523.1120	0.960285	521.3720
[-5 2]´	201.8490	0.946833	200.8020
[3 -1]´	82.09800	0.974124	82.01260
[7 2]´	419.3380	1.059350	417.1940
[9 1]´	712.8460	1.008720	712.7670
[9 3]´	727.4580	0.945638	723.9750
[3 8]´	615.4370	0.817115	586.6210

Table 4.10

Estimated Mean Squared Errors for $T_*(y_1)$ and $D_*(y_1)$

$P_1 = 1 \quad P = 1 \quad N = 10 \quad N_1 = 10 \quad n = 20 \quad q = 2 \quad K = 2$

$X_1' = [1 \;\; 2 \;\; -1 \;\; 3 \;\; 2 \;\; 0 \;\; -6 \;\; 9 \;\; 5 \;\; 0] \qquad W' = [1 \;\; 1 \;\; 1 \;\; 1]$

$$\nu' = [1 \;\; 1] \qquad \delta = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

THE MATRIX A	$MSE(T_*(Y_1))$	C^*	$MSE(D_*(Y_1))$
$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$	16.0789	1.003380	16.0782
$\begin{bmatrix} -2 & -1 \\ 0 & 2 \end{bmatrix}$	19.0045	0.988972	19.0012
$\begin{bmatrix} -4 & 6 \\ -5 & 2 \end{bmatrix}$	19.6352	1.018160	19.6271
$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$	15.4961	1.033170	15.4674
$\begin{bmatrix} 2 & -7 \\ 11 & 1 \end{bmatrix}$	19.5929	1.050400	19.3129
$\begin{bmatrix} 8 & -6 \\ -3 & 2 \end{bmatrix}$	18.6916	0.999554	18.6916
$\begin{bmatrix} 2 & 1 \\ 7 & 3 \end{bmatrix}$	18.9563	0.994723	18.9495

Table 4.11

Estimated Mean Squared Errors for $T_*(y_1)$ and $D_*(y_1)$

$P_1 = 1 \quad P = 1 \quad N = 10 \quad N_1 = 10 \quad n = 20 \quad q = 2 \quad K = 2$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad W' = [1\ 1\ 1\ 1] \qquad \nu' = [1\ 1]$$

$$\delta = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

X_1'	Results
$X_1' = [\ 1\ -1\ -1\ 3\ 2\ 3\ -2\ -1\ 0\ 0\ ]$	$MSE(T_*(Y_1)) = 24.77720$ $C^* = 0.968767$ $MSE(D_*(Y_1)) = 24.73680$
$X_1' = [\ .5\ .2\ .3\ .4\ .1\ 0\ 2\ -5\ 0\ 1\ ]$	$MSE(T_*(Y_1)) = 18.12610$ $C^* = 0.895501$ $MSE(D_*(Y_1)) = 17.71190$
$X_1' = [\ 5\ 4\ 1\ 6\ 7\ 2\ 5\ 3\ 1\ -11\ ]$	$MSE(T_*(Y_1)) = 22.64230$ $C^* = 0.973452$ $MSE(D_*(Y_1)) = 22.61050$
$X_1' = [\ 3\ 9\ -1\ -7\ 2\ -1\ -1\ -5\ 2\ 7\ ]$	$MSE(T_*(Y_1)) = 19.55000$ $C^* = 1.018360$ $MSE(D_*(Y_1)) = 19.53710$
$X_1' = [\ 1\ 2\ 3\ 4\ 5\ -7\ 0\ 0\ -1\ 0\ ]$	$MSE(T_*(Y_1)) = 22.61850$ $C^* = 1.014100$ $MSE(D_*(Y_1)) = 22.61080$
$X_1' = [\ 10\ 0\ -3\ 2\ 6\ 0\ 0\ 8\ -9\ 1\ ]$	$MSE(T_*(Y_1)) = 19.05700$ $C^* = 0.912588$ $MSE(D_*(Y_1)) = 18.75850$

Table 4.12

Estimated Mean Squared Errors for $T_*(y_1)$ and $D_*(y_1)$

$P_1 = 1 \quad P = 1 \quad N = 10 \quad N_1 = 10 \quad n = 20 \quad q = 2 \quad K = 2$

$X_1' = [1 \ \ 2 \ -1 \ \ 3 \ \ 2 \ \ 0 \ -6 \ \ 9 \ \ 5 \ \ 0] \qquad W' = [1 \ 1 \ 1 \ 1]$

$$\nu' = [1 \ 1] \qquad A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\delta = \begin{bmatrix} 1 & 2 & 7 & -5 & 1 & 0 & 0 & 1 & 1 & 1 & 3 & 1 & 1 & -7 & 1 & 1 & 0 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$MSE(T_*(Y_1)) = 85.28590$

$C^* = 0.871608$

$MSE(D_*(Y_1)) = 83.68190$

$$\delta = \begin{bmatrix} 1 & 2 & 7 & -5 & 1 & 0 & 0 & 1 & 1 & 1 & 3 & 1 & 1 & -7 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 2 & 0 & 3 & 1 & 2 & 5 & 4 & 3 & -1 & -1 & 0 & 0 & 4 & 3 & -1 & 2 & 0 & 2 & 0 \end{bmatrix}$$

$MSE(T_*(Y_1)) = 81.83200$

$C^* = 0.964057$

$MSE(D_*(Y_1)) = 81.68990$

$$\delta = \begin{bmatrix} -1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 2 & 3 & 0 & 3 & 0 & 1 & 2 & 3 & 0 & 0 & 1 & 5 \\ 0 & 2 & 0 & 3 & 1 & 2 & 5 & 4 & 3 & -1 & -1 & 0 & 0 & 4 & 3 & -1 & 2 & 0 & 2 & 0 \end{bmatrix}$$

$MSE(T_*(Y_1)) = 62.05020$

$C^* = 1.056250$

$MSE(D_*(Y_1)) = 61.61520$

$$\delta = \begin{bmatrix} -1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 2 & 3 & 0 & 3 & 0 & 1 & 2 & 3 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 3 & 2 & 1 & 0 & 5 & 0 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 2 & 1 \end{bmatrix}$$

$MSE(T_*(Y_1)) = 47.22480$

$C^* = 1.068330$

$MSE(D_*(Y_1)) = 46.77320$

Table 4.13

Estimated Mean Squared Errors for $L_*(y_1)$ and $\tilde{G}_*(y_1)$

$P_1 = 1 \quad P = 1 \quad N = 3 \quad N_1 = 3 \quad n = 6 \quad q = 2 \quad K = 2$

$X_1' = [\ 15 \quad 21 \quad 3\ ] \qquad W' = [1\ 1\ 1\ 1]$

$$\delta = \begin{bmatrix} 6 & 3 & 9 & 0 & 0 & 0 \\ 0 & 10 & 11 & 1 & -9 & 3 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & .2 \\ .3 & 7 \end{bmatrix}$$

THE VECTOR ν	$MSE(L_*(Y_1))$	$\hat{a}_*$	$MSE(\tilde{G}_*(Y_1))$
[0 0]′	.114496E+07	0.143629	17067.5
[1 0]′	.108948E+07	0.141500	15928.0
[.1 0]′	855835	0.146048	01297.9
[.2 .1]′	891419	0.142418	13206.3
[1 2]′	.100173E+07	0.144510	15228.9
[.5 .6]′	801110	0.141592	11864.7
[2.1 .7]′	865359	0.142975	12771.1
[0 -.1]′	948115	0.142866	14013.5
[0 -1]′	.105135E+07	0.145073	15797.3
[0 -3]′	.104923E+07	0.139902	16198.4
[-1 1]′	698203	0.139602	10135.7
[-1 2]′	872157	0.140836	13255.2
[-2 3]′	867841	0.142670	13418.0
[-1 -1]′	739628	0.143081	11070.3
[-.1 -.1]′	941910	0.144149	14009.0
[.05 .05]′	.100548E+07	0.142170	14893.3
[-.01-.01]′	.102840E+07	0.143461	15505.3
[2 -1]′	.103032E+07	0.142161	15337.2
[.9 .8]′	.104053E+07	0.145113	15681.9
[-.2 -.4]′	.118923E+07	0.145081	18019.2
[4 3]′	875047	0.149287	13238.9
[17 1]′	.348423E+07	0.141873	46920.1
[3 19]′	.126617E+07	0.187440	36870.8
[-.7 .8]′	.110101E+07	0.140744	16240.3

Table 4.14

Estimated Mean Squared Errors for $L_*(y_1)$ and $\breve{G}_*(y_1)$

$P_1 = 1$ $P = 1$ $N = 3$ $N_1 = 3$ $n = 6$ $q = 2$ $K = 2$

$X_1' = [\ 16\ \ 21\ \ 3\ ]$ $W_2' = [1\ \ 1]$ $\nu' = [1\ \ 1]$

$$\delta = \begin{bmatrix} 6 & 3 & 9 & 0 & 0 & 0 \\ 0 & 10 & 11 & 1 & -9 & 3 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & .2 \\ .3 & 7 \end{bmatrix}$$

THE VECTOR W_1	$MSE(L_*(Y_1))$	$\hat{a}_*$	$MSE(\breve{G}_*(Y_1))$
[1 2]′	.118367E+07	0.236483	056264.4
[1 3]′	.142565E+07	0.308810	124593.0
[1 5]′	.250903E+07	0.423722	427732.0
[2 3]′	.142661E+07	0.308291	123630.0
[2 4]′	.227591E+07	0.370364	292052.0
[−1 −1]′	504774	−0.124690	013056.0
[−1 5]′	.228325E+07	0.423301	390455.0
[−1 10]′	.373272E+07	0.588860	.12722E+07
[−5 2]′	.118983E+07	0.243252	060664.7
[3 −1]′	604537	−0.134643	018623.0
[7 2]′	.122365E+07	0.229913	055106.6
[9 1]′	974579	0.136443	012937.3
[9 3]′	.143534E+07	0.306309	121028.0
[3 8]′	.348167E+07	0.533862	971031.0

Table 4.15

Estimated Mean Squared Errors for $L_*(y_1)$ and $\breve{G}_*(y_1)$

$P_1 = 1 \quad P = 1 \quad N = 3 \quad N_1 = 3 \quad n = 6 \quad q = 2 \quad K = 2$

$X_1' = [\ 15 \quad 21 \quad 3\ ] \qquad W_1' = [1 \ 1] \qquad \nu' = [1 \ 1]$

$$\delta = \begin{bmatrix} 6 & 3 & 9 & 0 & 0 & 0 \\ 0 & 10 & 11 & 1 & -9 & 3 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & .2 \\ .3 & 7 \end{bmatrix}$$

THE VECTOR W_2	$MSE(L_*(Y_1))$	$\hat{a}_*$	$MSE(\breve{G}_*(Y_1))$
[1 2]´	.244355E+07	.892951E-01	11221.3
[1 3]´	.774780E+07	.611638E-01	16454.2
[1 5]´	.154587E+08	.474651E-01	13770.6
[2 3]´	.873281E+07	.660333E-01	18254.1
[2 4]´	.143368E+08	.548032E-01	18701.1
[-1 -1]´	589092	-.124421	15538.3
[-1 5]´	.164902E+08	.419336E-01	15601.4
[-1 10]´	.701502E+08	.297894E-01	32395.9
[-5 2]´	.293571E+07	.515685E-01	14863.1
[3 -1]´	390070	-.214445	12321.0
[7 2]´	.445135E+07	.110356	17852.1
[9 1]´	140064E+07	.184788	16686.6
[9 3]´	.879911E+07	.891274E-01	17907.7
[3 8]´	.496604E+08	.394884E-01	22606.7

Table 4.16

Estimated Mean Squared Errors for $L_*(y_1)$ and $\breve{\theta}_*(y_1)$

$P_1 = 1$ $P = 1$ $N = 3$ $N_1 = 3$ $n = 6$ $q = 2$ $K = 2$

$X_1' = [\ 15\ \ 21\ \ 3\ ]$ $W' = [1\ 1\ 1\ 1]$ $\nu' = [1\ 1]$

$$\delta = \begin{bmatrix} 6 & 3 & 9 & 0 & 0 & 0 \\ 0 & 10 & 11 & 1 & -9 & 3 \end{bmatrix}$$

THE MATRIX A	$MSE(T_*(Y_1))$	$\hat{a}_*$	$MSE(\breve{\theta}_*(Y_1))$
$\begin{bmatrix} 1 & .1 \\ .9 & 5 \end{bmatrix}$	.243304E+07	0.177769	65648.200
$\begin{bmatrix} 1 & -.8 \\ .9 & 9 \end{bmatrix}$	69557.60	0.327787E-01	01109.550
$\begin{bmatrix} 3 & -.6 \\ .7 & 10 \end{bmatrix}$	05490.57	0.402814	00536.924
$\begin{bmatrix} 3 & -.01 \\ .55 & 15 \end{bmatrix}$	.320838E+10	0.620365E-01	.125505E+08
$\begin{bmatrix} 6 & .4 \\ 1 & 8 \end{bmatrix}$	472649	0.110658	06317.730
$\begin{bmatrix} 11 & 2 \\ -1 & 5 \end{bmatrix}$	04133.58	0.102456	00930.010
$\begin{bmatrix} -13 & 4 \\ -2 & 19 \end{bmatrix}$	.116670E+08	0.729345E-01	68423.800

Table 4.17

Estimated Mean Squared Errors for $L_*(y_1)$ and $\breve{G}_*(y_1)$

$P_1 = 1 \quad P = 1 \quad N = 3 \quad N_1 = 3 \quad n = 6 \quad q = 2 \quad K = 2$

$W' = [1\ 1\ 1\ 1] \qquad \nu' = [1\ 1]$

$$\delta = \begin{bmatrix} 6 & 3 & 9 & 0 & 0 & 0 \\ 0 & 10 & 11 & 1 & -9 & 3 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & .2 \\ .3 & 7 \end{bmatrix}$$

THE MATRIX X_1	$MSE(L_*(Y_1))$	$\hat{a}_*$	$MSE(\breve{G}_*(Y_1))$
[1 -1 -6]′	626491	.682688E-01	08808.680
[.8 .2 .4]′	.148698E+07	.131142	24402.100
[11 -7 8]′	127016	.184200	02397.070
[-9 3 .8]′	40730.3	.197142	01279.980
[1 -8 .78]′	8884.87	.339142	00883.038
[10 0 -3]′	424862	.713796E-01	04028.600
[-1 -7 8]′	.113117E+07	.131888	18017.100
[20 -8 13]′	.608988E+08	.113470	772948.00

Table 4.18

Estimated Mean Squared Errors for $L_*(y_1)$ and $\tilde{G}_*(y_1)$

$P_1 = 1 \quad P = 1 \quad N = 3 \quad N_1 = 3 \quad n = 6 \quad q = 2 \quad K = 2$

$X_1' = [\ 18 \quad 21 \quad 3\ ] \qquad W' = [1 \ 1 \ 1 \ 1]$

$\nu' = [1 \ 1] \qquad A = \begin{bmatrix} 1 & .2 \\ .3 & 7 \end{bmatrix}$

$\delta = \begin{bmatrix} 4 & 3 & 7 & 0 & 0 & -6 \\ 0 & 10 & 11 & 1 & -9 & 3 \end{bmatrix}$	$MSE(L_*(Y_1))=$ .168718E+09 $\hat{a}_* =$ .977781E-01 $MSE(\tilde{G}_*(Y_1))=$ .160319E+07
$\delta = \begin{bmatrix} 4 & 3 & 7 & 0 & 0 & -6 \\ 2 & 7 & 1 & 0 & -3 & 8 \end{bmatrix}$	$MSE(L_*(Y_1))=$ .148322E+09 $\hat{a}_* =$ .982028E-01 $MSE(\tilde{G}_*(Y_1))=$ .143408E+07
$\delta = \begin{bmatrix} 0 & 2 & 0 & 3 & 1 & 2 \\ 8 & -1 & 11 & 7 & 0 & .9 \end{bmatrix}$	$MSE(L_*(Y_1))=$ 78286.8 $\tilde{a}_* =$ .148048 $MSE(\tilde{G}_*(Y_1))=$ 1168.22
$\delta = \begin{bmatrix} 3 & 9 & -1 & -8 & 2 & -1 \\ -1 & 0 & .7 & 4 & -.6 & 8 \end{bmatrix}$	$MSE(L_*(Y_1))=$ .686937E+07 $\hat{a}_* =$ .106482 $MSE(\tilde{G}_*(Y_1))=$ 74664.6

4.3 Conclusions.

- The shrinkage estimators proposed in Chapter 3 dominates the estimators developed in Chapter 2 in the sense of MSE.

- Our numerical results indicate that the estimated MSE's of $\breve{\theta}_*(y_1)$ when the shrinkage factor is replaced by its sample estimate are still smaller than those of $L_*(y_1)$.

- Our simulation study supports that we get a better reduction in the MSE using shrinkage approach when ν is unknown than the case when ν is known.

- Since there is a loss of information when ν is unknown, the MSE's of $L_*(Y_1)$ are relatively large as compared with the case when ν known.

- The gain in using the shrinkage estimators is observed in terms of the estimated MSE. This have been achieved at the cost of bias. Our shrinkage estimators although have smaller estimated MSE but they are biased.

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ملخص

التنبوء بالمشاهدات المستقبلية وتقدير المعلمات في النماذج الاحتمالية الانحدارية

في هذه الاطروحة تناولنا تقدير الاقترانات المعلمية في النماذج الاحتمالية الانحدارية والتنبوء بالمشاهدات المستقبلية المرتبطة احتماليا بمجموعة من المتغيرات اللاعشوائية كذلك اوجدنا المقدرات الغير منحازه صاحبة اقل وسط مربع الخطأ والمقدرات المتقلصة للمعلمات التي تهمنا ثم درسنا خصائص هذه المقدرات رياضيا وعن طريق المحاكاه . يتضح لنا من المحاكاه التي قمنا ان المقدرات المتقلصه افضل من المقدرات الغير منحازه وذلك في وسط مربع الخطا .